BOUNDARY IMAGES OF MEROMORPHIC FUNCTIONS

BY
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1. Introduction.

1.1. Let D be the open unit disk, bdy D its bounding circle, and S the unit sphere. If $f:\bar{D}\to S$ is continuous, then $f(\text{bdy } D)$ is a Peano continuum. (We remind the reader that a Peano continuum is a locally connected [metric] continuum, and that it is characterized as the continuous image of a closed interval.)

It has been shown (by G. R. MacLane [8]; Piranian, Titus, and Young [12]; Salem and Zygmund [16]; and Schaeffer [17]) that there exist functions f, continuous on the closed disk $\bar{D}$ and analytic on D, for which $f(\text{bdy } D)$ is a solid square. Indeed Rudin [14] has proved the following theorem: Suppose (a) E is a closed subset of the circle bdy D, E having Lebesgue measure zero; (b) ϕ is a continuous function on E into the complex plane; (c) T is a two-cell such that $\phi(E)\subset T$. Then there exists a function f analytic on D and continuous on $\bar{D}$ such that (i) $f(z)=\phi(z)$ for all z in E; (ii) $f(\bar{D})\subset T$. Thus, given any compact set C in the complex plane, there is a continuous function ϕ of the Cantor set $E\subset \text{bdy } D$ onto C [25, p. 35, (4.7)], and $C\subset f(\text{bdy } D)$ for the f given above.

The author [4] has shown that there is a Peano continuum in the plane which is not the image of the circle bdy D for any function meromorphic in D and continuous in $\bar{D}$. A purely topological characterization of those Peano continua which are boundary images is given here.

THEOREM. *A nonempty Peano continuum P on S satisfies either both or neither of the following conditions:*

(C_1) *There exists a map $f:\bar{D}\to S$, meromorphic on D, for which $P=f(\text{bdy } D)$.*

(C_2) *There exists a finite or countably infinite family of simply connected regions U_n $(n=1,2,\cdots)$ such that:*

(a) *P is the closure of $\bigcup_n \text{bdy}(U_n)$;*

(b) *for each $n>1$, there exists $m<n$ for which $U_n\cap U_m\neq\emptyset$ and*

$$\text{bdy}(U_n)\cap\text{bdy}(U_m)\neq\emptyset;$$

and, if the family is infinite,

(c) $\limsup(U_n)\subset P$.

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The *limit superior* of $\{U_n\}$ is the set of all points x in S such that every neighborhood of x contains points of infinitely many regions U_n. If (c) is replaced by "$\operatorname{diam}(U_n) \to 0$," the resulting condition is equivalent (see 3.3 and 3.6). Condition C_1' is defined from C_1 by replacing "meromorphic" by "analytic"; Condition C_2' is defined from C_2 by adding the requirement that no region U_n contain infinity. *If P is a nonempty Peano continuum on S, other than a single point, then P satisfies C_1' if and only if it satisfies C_2'.*

Condition C_2 has a degree of naturalness. Given any Peano continuum P on S, each of the countable number of components V_m of $S - P$ is a simply connected region (indeed, its boundary is locally connected [25, p. 106, (2.2)]), and the diameters converge to zero [25, p. 113, (4.4)]. There is a countable family of disks D_m in $\operatorname{int} P$, such that the bounding circles are dense in $\operatorname{int} P$ and $\operatorname{diam}(D_m) \to 0$. The sets V_m and D_m, together, constitute regions U_n satisfying Condition C_2 except for (b). In "most" cases, these sets can be modified to satisfy (b).

For example, suppose that P is a Peano continuum such that $\dim P = 1$ and $S - P$ has a finite number of components. Let V_m $(m = 1, 2, \cdots, k)$ be the components of $S - P$, and let v_m be points on $\operatorname{bdy}(V_m)$. Let γ_m $(m = 1, 2, \cdots, k-1)$ be arcs [24, p. 81, (3.11)] in P from v_m to v_{m+1} $(\gamma_m = \{v_m\}$ if $v_m = v_{m+1})$, let $U_{2m-1} = V_m$ $(m = 1, 2, \cdots, k)$ and $U_{2m} = S - \gamma_m$ $(m = 1, 2, \cdots, k-1)$. Then P satisfies C_2, and thus C_1.

While the example of [3, p. 52, Remark] has dimension one, its complement has an infinite number of components, and it fails to satisfy C_1 and C_2. If $\{U_n\}$ is any sequence satisfying (a) and (b) of C_2, then an infinite number of regions U_n will contain the unbounded complementary component; thus the example will not satisfy C_2 (c).

Also the universal plane curve and the triangular curve of Sierpiński [7, pp. 202–203] both satisfy C_2 and thus C_1. In the latter case, using the notation of [7], the regions U_n in order are: T, T_0, T_1, T_2, T_{00}, T_{01}, etc.

In a different but related direction, Morse [9, p. 74], and Titus [21; 22] have given conditions for a function (i.e., a *parametrized* curve) to be a boundary function for a function f analytic on D, and continuous on $\bar{D}$ (see §5).

1.2. Given a meromorphic function $f: D \to S$, the *global cluster set* $C(f)$ is defined as follows: $y \in C(f)$ if there exists a sequence of points z_n in D such that $\lim |z_n| = 1$ and $\lim f(z_n) = y$. The set $C(f)$ is a continuum, and, of course, if f is continuous on $\bar{D}$, then $C(f) = f(\operatorname{bdy} D)$. In [5, p. 123], Collingwood and Cartwright asked whether every continuum $C \subset S$ could arise as the global cluster set of some meromorphic function. Independently D. B. Potyagaĭlo [13] and W. Rudin [15] gave a counter-example, which is not a Peano continuum. In [4] the author gave the Peano continuum example mentioned earlier, and a sufficient condition. (The condition is not equivalent, even for Peano continua, to a sufficient condition given by Potyagaĭlo in [13].) For Peano continua that condition [4, p. 53] is almost identical to, and *is* equivalent to (see 3.6) C_2. We now observe that for the class of Peano continua at least, the condition is also necessary.

THEOREM. *For Peano continua $P \neq \emptyset$, Condition C_2 is equivalent to Condition C_3: There exists a function f meromorphic on D for which P is the global cluster set $C(f)$.*

Condition C_3' (infinity not in the range of f) is equivalent to C_2', except for single points.

The following condition, C_4, implies the denial of C_3 for (arbitrary) continua P.

There exist components K_i $(i = 1,2,\cdots,I)$ of $S - P$ and a sequence $\{R_n\}$ of regions such that, for each n $(n = 1,2,\cdots)$:

(a) $R_n \neq S$,

(b) $P \cap R_n \neq \emptyset$,

(c) $\text{bdy}(R_n) \subset \bigcup_{i=1}^{I} \text{bdy}(K_i)$, *and*

(d) *each open set W which meets* $\text{bdy}(R_n)$, *contains a region R_m having $m > n$.*

The counter example of [4, §2] satisfies this condition where $I = 1$, $K_1 = S - P$, and the regions R_n are the (open) disks. The proof that Condition C_4 implies the denial of C_1 is a generalization of that proof.

Proof. Suppose that there exists a continuum P satisfying both C_3 and C_4. Then P is not a single point, so that $f(D)$ is open. Let C be the set of points in D at which f is not one-to-one. Let B_i be $K_i - f(C)$, and let E_i be $f^{-1}(B_i)$ $(i = 1,2,\cdots,I)$. As in [4, p. 52], if $E_i \neq \emptyset$, then E_i is a covering space of the base space B_i with projection map f and degree n_i, $1 \leqq n_i < \infty$. (If $E_i = \emptyset$, let $n_i = 0$.)

For each ε, $0 < \varepsilon < 1$, let A_ε be the open annular region in D at distance less than ε from the circle bdy D. By (a) and (d) there exists a natural number $n(1)$ such that diam $(R_{n(1)}) <$ diam P. Since P satisfies Condition C_3, diam $P \leqq$ diam $(f(D))$, and by (b) $f(D)$ meets $R_{n(1)}$. Thus $f(D)$ meets $\text{bdy}(R_{n(1)})$. As a result, there is an $\varepsilon(1) > 0$ and a set U_1 open in $D - A_{\varepsilon(1)}$ such that $\text{diam}(f(U_1)) < \text{diam}\, P$ and $f(U_1)$ meets $\text{bdy}(R_{n(1)})$.

In general, suppose that open sets $U_j \subset D$, an integer $n(k)$, and $\varepsilon(k) > 0$ have been given such that the sets U_j are mutually disjoint, $f(U_j) \subset f(U_{j-1})$ $(j = 1,2,\cdots,k;\ U_0 = D)$, $\bigcup_{j=1}^{k} U_j$ does not meet $A_{\varepsilon(k)}$, and $f(U_k)$ meets $\text{bdy}(R_{n(k)})$. By (d) there exists $n(k+1)$ such that $R_{n(k+1)} \subset f(U_k)$. By (b) $f(A_{\varepsilon(k)})$ meets $R_{n(k+1)}$, and thus meets $\text{bdy}(R_{n(k+1)})$. Hence there is an $\varepsilon(k+1) > 0$ and an open set $U_{k+1} \subset A_{\varepsilon(k)} - A_{\varepsilon(k+1)}$ such that $f(U_{k+1}) \subset f(U_k)$ and $f(U_{k+1})$ meets $\text{bdy}(R_{n(k+1)})$.

For each k $(k = 1,2,\cdots)$, there exists a component K_i $(i = 1,2,\cdots,I)$ such that $f(U_k)$ meets K_i (by (c)). Thus $n_i \geqq k$, and a contradiction results.

1.3. REMARK. Let g be a function analytic on D and continuous on $\bar{D}$ such that $g(\text{bdy}\, D) = \bar{D}$ (given, for example, by [12]). Suppose that $g \not\subset (\bar{D})\bar{D}$. Since infinity is not in $g(\bar{D})$, $\text{bdy}(g(\bar{D}))$ meets $S - \bar{D}$, say in q. Since $q \notin \bar{D}$, $q \notin g(\text{bdy}\, D)$.

Thus $q \in g(D)$. Since $g(D)$ is open, we have a contradiction. Thus $g(\bar{D}) \subset \bar{D}$, so that $g(D) \subset D$.

Let E be the upper half-plane, considered as a topological disk on S, and let h be a homeomorphism of $\bar{D}$ on $\bar{E}$ (closure in S), h analytic on D. The function f defined by $f(z) = (h(g(z)))^2$ is analytic on D, and $f(\operatorname{bdy} D) = S$; thus, S satisfies Condition C_1'. (This conclusion could have been proved from results in this paper, without the use of [12].)

Let $\mathfrak{D}_j$ be the set of all open squares with sides parallel to the real and imaginary lines, sides of length 2^{-j+1}, and centers $(m + ik)2^{-j}$ $(m, k = 1, 3, \cdots, 2^j - 1;$ $j = 1, 2, \cdots)$. The squares of $\mathfrak{D}_j$, taken in order, constitute a sequence $\{V_n\}$ of regions for the unit square $\{x + iy : 0 \leqq x, \ y \leqq 1\}$ satisfying C_2'. Now $S = D_1 \cup D_2$, where D_1 and D_2 are closed topological disks. Let $p \in D_1 \cap D_2$, and let h_t be a homeomorphism of the unit square onto D_t $(t = 1, 2)$. The sequence $\{U_n\}$ of regions defined by $U_1 = S - \{p\}$, $U_{2n} = h_1(V_n)$, and $U_{2n+1} = h_2(V_n)$ $(n = 1, 2, \cdots)$ satisfies Condition C_2' for S.

Also, a single point satisfies both C_1 and C_2. Thus, to prove 1.1, it is sufficient to assume hereafter that *P is a fixed nonempty Peano continuum, neither S nor a single point.*

The outline of the proof is given now. After preliminary lemmas (§2), it is shown (3.1) that each of Conditions C_2 and C_3 implies a certain condition. This condition implies (3.2, 3.5, 3.6) the existence of a chain (3.3), which is a family of regions U_n satisfying the properties of C_2 and some others. As a result, Condition C_3 implies C_2. In 4.2 it is proved that the existence of a chain implies C_1, except that f is only light open. The existence of the meromorphic function is given in 4.3, using the extension of Stoilow's Theorem given in [3], so that C_2 implies C_1. Clearly, C_1 implies C_3; thus, the three conditions are equivalent, as are C_1', C_2', and C_3'. The special case in which the function of C_1 is a homeomorphism on D is treated in 2.1, and extensions are discussed in §5.

The closure of X will be denoted either by $\mathrm{Cl}[X]$ or by $\bar{X}$; the interior of X, by $\operatorname{int} X$; the distance between two points p and q on S, by $d(p, q)$; the set of points of distance from p at most ε, by $S(p, \varepsilon)$; and the null set by $\emptyset$. The term "map" will mean "continuous function."

2. **Preliminary results.** It is well known [1, p. 86] that if f is a conformal homeomorphism of the open unit disk D onto the interior X of a closed topological disk, then there is an extension of f to a homeomorphism of $\bar{D}$ onto $\bar{X}$. The following result generalizes this theorem.

2.1. Theorem. *If B is a simply connected region on S such that* bdy B *is a nondegenerate Peano continuum, then each prime end is a single point. Thus, if f is any conformal homeomorphism of D onto B, then f can be extended to $\bar{D}$ to be continuous.*

Proof. We use the terminology of Piranian [11, pp. 45–46]; the term "chain" here should not be confused with that of 3.3. Suppose that there exists a prime end E whose impression $I(E)$ contains (at least) two distinct points p and q; we may suppose that p is a principal point. The equivalence class E contains a chain $\{c_n\}$ such that each $c_n \in S(p, 1/n)$ and the sets B_n are nested. (For each crosscut c_n, B_n is that component of $B - c_n$ which contains c_{n+1}; the impression $I(E)$ is $\bigcap_n \bar{B}_n$.) Let $\{p_n\}$ and $\{q_n\}$ be sequences of points such that each $p_n \in c_n \cap B$, $p_n \to p$, each $q_n \in B_{n-1} - B_n$, and $q_n \to q$. Let $\{\gamma_n\}$ be a sequence of arcs in B, each γ_n joining q_{n-1} to q_n and meeting c_n only in p_n. (Thus γ_n passes "through" c_n.) Choose $\varepsilon > 0$ such that neither q nor any q_n is in $S(p, \varepsilon)$. Since bdy B is locally connected, $S(p, \varepsilon) \cap \text{bdy}\, B$ contains a connected neighborhood N of p in bdy B; there exists a natural number n such that

$$S(p, 1/n) \cap \text{bdy}\, B \subset N.$$

The arc γ_n separates $S(p, \varepsilon)$ into at least two components, one containing p, another containing an endpoint r of c_n, with r in bdy $B \cap S(p, 1/n)$. The connectivity of N is thus contradicted, so that $I(E)$ must be $\{p\}$.

The second statement follows from Carathéodory's principal theorem on prime ends [2, p. 350].

2.2. Definition. Let U be S, or D where bdy $D \subset P$. Let $\varepsilon > 0$, and let K_i $(i = 1, 2, \cdots, I)$ be those components of $U - P$ having diam$(K_i) \geqq \varepsilon/3$ (there are only a finite number of them [25, p. 113, (4.4)]). An *ε-region* (for P and U) is a component of $U - \bigcup_{i=1}^{I} \bar{K}_i$.

Remarks. If V is a region in U, bdy $V \subset P$, and diam $V \leqq \varepsilon/3$, then V is contained in some ε-region R. Since $P \cap \bar{D}$ is a Peano continuum [25, p. 112, (4.2) and p. 113, (4.4)], the ε-regions for P and D are the ε-regions for $P \cap \bar{D}$ and S, except possibly for $S - \bar{D}$. Thus, it suffices to prove 2.3, 2.4, 2.5, and 2.6 for $U = S$.

2.3. Lemma. *For each $\varepsilon > 0$, an ε-region R has Property* S [25, *p.* 20], *and thus $\bar{R}$ is a Peano continuum.*

Proof. Let X_j $(j = 1, 2, \cdots, J)$ be the components of $\bigcup_{i=1}^{I} \bar{K}_1$. Since each bdy (K_i) is locally connected [25, p. 106, (2.2)], each K_i has Property S [25, p. 109, (3.2)], and $\bar{K}_i$ does also [25, p. 20, (15.3)]; thus, each X_j has Property S and is locally connected. Let S_j be that component of $S - X_j$ containing R $(j = 1, 2, \cdots, J)$; then S_j has Property S [25, p. 106, (2.2) and p. 109, (3.2)].

Let δ be any positive number less than one half the minimum distance from any of the sets bdy(S_j) to any other of them (since bdy$(S_j) \subset X_j$, $\delta > 0$; $j = 1, 2, \cdots, J$). Each S_j is the union of a finite family $\mathfrak{Y}$ of connected subsets, each of diameter less than δ; let Y_k $(k = 1, 2, \cdots, N)$ be those subsets (for all j) contained in R. We will show that $R \subset \bigcup_{k=1}^{N} Y_k$. Let $p \in R$. If $d(p, \text{bdy}(S_j)) \geqq \delta$, for all j, then any connected subset $Y \in \mathfrak{Y}$ containing p is contained in R; thus it is a set Y_k. If $d(p, \mathrm{bdy}(S_j)) < \delta$, for some j, then $p \in Y$, where $Y \in \mathfrak{Y}$ is a connected subset for S_j. Since $\mathrm{diam}\, Y < \delta$, Y is disjoint from $\mathrm{bdy}(S_i)$, for $i \neq j$; as a result, Y is a set Y_k. Thus, R has Property S. Since $\bar{R}$ also has Property S, it is locally connected [25, p. 20, (15.1) and (15.3)].

2.4. LEMMA. *If, for some* $\varepsilon > 0$, *there are an infinite number of distinct* ε-*regions* R_n $(n = 1, 2, \cdots)$, *then* $\mathrm{diam}(R_n) \to 0$.

Proof. We first prove that if R_1 and R_2 are two (ordinary) regions on S such that R_1 is simply connected and

$$\mathrm{bdy}(R_1) \cap \mathrm{bdy}(R_2) = \emptyset,$$

then $R_1 \cap R_2$ is a (possibly empty) region. We may suppose that $R_1 \cap R_2$ is neither $\emptyset$ nor R_2. If C is any component of $R_1 \cap R_2$, then there is an arc γ in R_2 joining a point of C to a point of $R_2 - (R_1 \cap R_2)$. As a result, $\mathrm{bdy}\, C$ meets $R_2 \cap \mathrm{bdy}(R_1)$, and since $\mathrm{bdy}(R_1)$ is connected, $\mathrm{bdy}(R_1) \subset R_2$. Thus there is a neighborhood N of $\mathrm{bdy}(R_1)$ (in S) such that $N \cap R_1 \subset C$; since C is an arbitrary component of $R_1 \cap R_2$, $C = R_1 \cap R_2$; i.e., $R_1 \cap R_2$ is connected.

Let X_j $(j = 1, 2, \cdots, J)$ be the components of $\bigcup_{i=1}^{I} \bar{K}_i$ (cf. 2.2); then, as in the proof of 2.3, each X_j is a Peano continuum. Let $\delta > 0$ be given; it is desired to prove that only a finite number of ε-regions have diameter at least δ.

Let R be any ε-region with $\mathrm{diam}\, R \geqq \delta$; since R is a component of $S - \bigcup_{j=1}^{J} X_j$, it is a component of $\bigcap_{j=1}^{J} R_{j,k(j)}$, where each set $R_{j,k(j)}$ is a component of $S - X_j$, and $\mathrm{diam}(R_{j,k(j)}) \geqq \delta$. From the conclusion of the first paragraph and induction, this intersection set is connected and thus *is* R. Since only a finite number of components of $S - X_j$ have diameter at least δ [25, p. 113, (4.4)], the conclusion follows.

2.5. LEMMA. *If* R *is an* ε-*region, and*

$$\mathrm{diam}\, R < \varepsilon < \frac{1}{2}\ \mathrm{diam}\, S,$$

then R *is contained in a closed topological disk* E *where* $\mathrm{int}\, E \subset U$, $\mathrm{diam}\, E = \mathrm{diam}\, R$ *and* $\mathrm{bdy}\, E \subset P$. *In fact, if* $\mathrm{diam}\, R < \varepsilon/3$, *then* $\bar{R} = E$.

Proof. First, suppose that $U = S$. Let $p \in R$; then $R \subset S(p, \varepsilon)$. If W is that component of $S - \bar{R}$ which contains $S - S(p, \varepsilon)$, then $\mathrm{bdy}\, W$ is locally connected (2.3 and [25, p. 106, (2.2)]) and thus contains a circle C separating W from R [24, p. 114, (6.7)]. Let E be the closure of the component of $S - C$ containing R. Since $C \subset \mathrm{bdy}\, R$, $\mathrm{diam}\, C \leqq \mathrm{diam}\, R$. But $\mathrm{diam}\, E = \mathrm{diam}\, C$. On the other hand, since $R \subset E$, $\mathrm{diam}\, R \leqq \mathrm{diam}\, E$.

Suppose $\mathrm{diam}\, R < \varepsilon/3$. Then E does not contain any K_i $(i = 1, 2, \cdots, I)$, and thus is disjoint from them. Since $\mathrm{int}\, E$ is a region it is contained in R; therefore, it is R.

If $U = D$, then R is an ε-region for S and $P \cap \bar{D}$ (by 2.2); thus, if E is the closed topological disk given by the above proof, then bdy $E \subset P \cap \bar{D}$. Since $E \cap R \neq \emptyset$, $\operatorname{int} E \subset D$.

2.6. LEMMA. *If R is an ε-region, $\varepsilon < \operatorname{diam} R$, and $\varepsilon < \frac{1}{2}$ diam S, then there are a finite number of closed topological disks E_m $(m = 1, 2 \cdots, M)$ such that*

(1) $R \subset \bigcup_{m=1}^{M} \operatorname{int}(E_m)$,

(2) *each* $\operatorname{int}(E_m) \subset U$ *and* $\operatorname{bdy}(E_m) \subset P$, *and*

(3) $\operatorname{diam}(E_m) < \varepsilon$.

Proof. We may suppose that $U = S$; the proof in case $U = D$ follows as in 2.5. By 2.3, R has Property S. Thus, it is the union of a finite number of open sets, each having diameter less than $\varepsilon/3$ and having Property S [25, p. 21, (15.41)]. Let Y be one of these sets; it suffices to construct a closed disk E containing Y and also satisfying (2) and (3).

Let V_n $(n = 1, 2, \cdots)$ be the components of $S - P$ which meet Y, and let Z be the region $Y \cup \bigcup V_n$. Since each $V_n \subset R$ and each diam $(V_n) < \varepsilon/3$ (by the definition of ε-region), diam $Z < \varepsilon$. Let $\delta > 0$; $\bar{Y}$ is the union of a finite number of connected sets A_j, where $\operatorname{diam}(A_j) \leqq \delta/3$ $(j = 1, 2, \cdots, J)$. There is a natural number N such that, for all $n > N$, $\operatorname{diam}(V_n) < \delta/3$ [25, p. 113, (4.4)]. Let B_j be the union of A_j with all the $\bar{V}_n$ it meets $(n > N; j = 1, 2, \cdots, J)$. Each set B_j is connected, and $\operatorname{diam}(B_j) < \delta$. Since

$$\operatorname{Cl}\left[\bigcup_{n=1}^{\infty} V_n\right] - \bigcup_{n=1}^{\infty} \bar{V}_n \subset \bar{Y},$$

$\bar{Z}$ is the union of the sets B_j $(j = 1, 2, \cdots, J)$ and $\bar{V}_n$ $(n = 1, 2, \cdots, N)$. Because δ is an arbitrary positive number, $\bar{Z}$ has Property S and thus is locally connected.

Since diam $\bar{Z} < \varepsilon$, for any p in $\bar{Z}$, $\bar{Z} \subset S(p, \varepsilon)$. Let W be the component of $S - \bar{Z}$ containing $S - S(p, \varepsilon)$. Since bdy W is locally connected and separates W from Z, it contains a circle C which also separates W from Z [24, p. 114, (6.7)]. Let E be the closure of the open topological disk of $S - C$ containing Z. Since bdy $E \subset \bar{Z}$, diam $\bar{Z} \geqq$ diam E; $\varepsilon < \frac{1}{2}$diam S, and (3) is satisfied. Suppose that there is a point $z \in (\operatorname{bdy} Z) - P$. Since $V_n \subset Z$ $(n = 1, 2, \cdots)$, z is in a component of $S - P$ which does not meet Y (and thus Z), contradicting the fact that $z \in \operatorname{bdy} Z$. Thus bdy Z $\subset P$. Since bdy $E \subset$ bdy Z, (2) is also satisfied.

2.7. DEFINITION. Two simply-connected regions U and V on S *meet properly* if there is an arc γ ending in a point p on bdy $U \cap$ bdy V, such that $\gamma - \{p\} \subset U \cap V$.

If bdy U is locally connected (but not a point), then there is a naturally *associated prime end* E of U whose impression $I(E)$ [11, p. 45] is $\{p\}$. Let ε be less than diam U, $\frac{1}{2}$diam(bdy U), and $\frac{1}{2}$diam γ. There is a subarc γ_n of γ such that $p \in \gamma_n$, the other endpoint q_n of γ_n is on the circle bdy$(S(p, \varepsilon/n))$, and $\gamma_n - (\{p\} \cup \{q_n\})$

is disjoint from that circle ($n = 1, 2, \cdots$). Let λ_n be the closure of the component of $U \cap \mathrm{bdy}(S(p, \varepsilon/n))$ containing q_n; since bdy $(S(p, \varepsilon/n))$ meets bdy U, λ_n is a crosscut. The family $\{\lambda_n\}$ defines a prime end E, and by (2.1) its impression $I(E)$ is $\{p\}$.

There is a conformal homeomorphism of D onto U, and it has a continuous extension f to $\bar{D}$ (by 2.1). The set

$$\mathrm{Cl}\,[f^{-1}(\gamma - \{p\})] \cap \mathrm{bdy}\, D$$

is connected (e.g., from [25, p. 14, (9.1)]). If it were not a point, then f would be constant along an arc, contradicting a slight modification of the Riesz-Nevanlinna Theorem [10, p. 19]. Thus $\mathrm{Cl}\,[f^{-1}(\gamma - \{p\})]$ is an arc μ meeting bdy D in a single point O, which corresponds to the prime end E under the Carathéodory theorem [2, p. 350]; in fact, $f(O) = p$.

2.8. Lemma *Let U and V be simply connected proper subregions of S such that* bdy U *and* bdy V *are locally connected, $U \cap V \neq \emptyset$, and* bdy $U \cap$ bdy $V \neq \emptyset$. *Then U and V meet properly.*

Proof. If $U \subset V$ or $V \subset U$, the conclusion follows from [25, p. 112, (4.2)]. Thus, we may suppose that there exist points $q \in U \cap V$, $r \in U - V$, and $s \in V - U$. Since each of bdy U and bdy V has Property S, their union has it also; thus it is locally connected. Let W be the component of

$$S - (\mathrm{bdy}\, U \cup \mathrm{bdy}\, V)$$

containing q; bdy W is locally connected [25, p. 106, (2.2)]. There is an arc α joining q to r in U; since $\alpha \not\subset W$, α meets bdy W, and thus bdy W meets bdy V. Similarly, bdy W meets bdy U. Since the union of the two closed sets bdy U and bdy V contains bdy W, which is connected, they have a common point p. There is an arc γ ending at p, $\gamma - \{p\} \subset W$; as a result, U and V meet properly.

3. Chains.

3.1. Lemma. *Let P satisfy either Condition C_2 or C_3, and let U be S, or D where* bdy $D \subset P$; *suppose that $U \not\subset P$. Let K be a component of $U - P$; let $\varepsilon > 0$, ε less than the minimum of* diam K, diam P, *and* $\frac{1}{2}$ diam S; *and let R_m ($m = 1, 2, \cdots$) be a collection of distinct ε-regions (for P and U) such that* $\mathrm{diam}(R_m) < \varepsilon/3$, $P \cap R_m \neq \emptyset$, *and* $\mathrm{bdy}(R_m) \cap \mathrm{bdy}\, U$ *contains no arc.*

Then, there is a natural number M and, corresponding to the regions R_m, two families of nonempty open sets X_m and Y_m such that:

(a) $\bar{Y}_m \subset X_m \subset U$;

(b) $Y_m \cap \mathrm{bdy}(R_m) \neq 0$;

(c) *if $j < m$, then either $X_m \subset X_j$ or $X_m \cap \bar{Y}_j = \emptyset$; and*

(d) *if $j(k)$ ($k = 1, 2, \cdots, m$) is a function such that $j(k) < j(k+1)$ and $X_{j(k+1)} \subset X_{j(k)}$ ($k = 1, 2, \cdots, m-1$), then $m \leqq M$.*

Proof. There are, actually, two separate proofs. First, suppose that P satisfies Condition C_3. Let K_i $(i = 1,2,\cdots,I)$ be as in 2.2 for ε, P, and U; then K is a region K_i. If Z is the set of zeros of the derivative of f in D, then $f(Z) \cap K_i$ has no limit point in K_i, since K_i is disjoint from the global cluster set $C(f) = P$. The restriction of f to $f^{-1}(K_i - f(Z))$ is a covering map (as in [4, p. 52]), and, for each q in $K_i - f(Z)$, $f^{-1}(q)$ has the same (finite) number M_i of points [18, p. 67]; let M be the maximum of the M_i $(i = 1,2,\cdots,I)$.

Let A_n be the annular region defined by $1 - (1/n) < |z| < 1$ $(n = 1, 2, \cdots)$. Since $P \cap R_1 \neq \emptyset$, $f(A_1) \cap R_1 \neq \emptyset$; but, since

$$\operatorname{diam}(R_1) < \operatorname{diam} P \leqq \operatorname{diam}(f(A_1)),$$

(A_1) meets $S - \bar{R}_1$. Thus, there exists a set W_1 open in A_1, such that $\bar{W}_1 \subset A_1$ and the open set $f(W_1)$, call it X_1, meets $\operatorname{bdy}(R_1)$. By 2.5 each set $\bar{R}_m$ is a closed topological disk, and by hypothesis $\operatorname{bdy}(R_m) \cap \operatorname{bdy} U$ contains no arc. Hence, we may suppose that $X_1 \subset U$. There exists an open set Y_1 such that Y_1 meets $\operatorname{bdy}(R_1)$, and $\bar{Y}_1 \subset X_1$.

There also exists a natural number $n(2)$ such that $\bar{W}_1 \cap A_{n(2)} \neq \emptyset$, and, in $A_{n(2)}$, an open set T such that $\bar{T}$ is compact, $f(T)$ meets $\operatorname{bdy}(R_2)$ and $f(T) \subset U$. There is an open set $W_2 \subset T$ such that the open set $f(W_2)$, call it X_2, also meets $\operatorname{bdy}(R_2)$, and either $X_2 \subset X_1$ or $X_2 \cap \bar{Y}_1 = \emptyset$. Let Y_2 be any open set such that Y_2 meets $\operatorname{bdy}(R_2)$ and $\bar{Y}_2 \subset X_2$.

Using finite induction, suppose that we have defined:

(1_m) mutually disjoint open sets W_j such that $\bar{W}_j \subset D$;

(2_m) X_j as $f(W_j)$ with $X_j \subset U$; and

(3_m) open sets Y_j such that $\bar{Y}_j \subset X_j$, Y_j meets $\operatorname{bdy}(R_j)$, and, if $i < j$, then either $X_j \subset X_i$ or $X_j \cap \bar{Y}_i = \emptyset$ $(j = 1,2,\cdots,m)$.

There exists a natural number $n(m+1)$ such that

$$A_{n(m+1)} \cap \bigcup_{j=1}^{m} \bar{W}_j = \emptyset,$$

and, in $A_{n(m+1)}$, an open set T such that $\bar{T}$ is compact, $f(T)$ meets $\operatorname{bdy}(R_{m+1})$, and $f(T) \subset U$. There is an open set $W_{m+1} \subset T$, such that $f(W_{m+1})$, call it X_{m+1}, meets bdy (R_{m+1}), and, for each j $(j = 1, 2, \cdots, m)$, either $X_{m+1} \subset X_j$ or $X_{m+1} \cap \bar{Y}_j = \emptyset$. Let Y_{m+1} be any open set such that Y_{m+1} meets $\operatorname{bdy}(R_{m+1})$ and $\bar{Y}_{m+1} \subset X_{m+1}$. Then W_j, X_j, and Y_j $(j = 1, 2, \cdots, m+1)$ satisfy (1_{m+1}), (2_{m+1}), and (3_{m+1}).

Thus, there exist families of sets X_m and Y_m satisfying conclusions (a), (b), and (c). For (d), suppose on the contrary, that there exist increasing natural numbers $j(k)$ $(k = 1,2,\cdots, M+1)$ such that each set $X_{j(k+1)} \subset X_{j(k)}$. By (a) and (b), $X_{j(M+1)}$ meets $\operatorname{bdy}(R_{j(M+1)})$, and thus meets one of the components K_i $(i = 1,2,\cdots,I)$. Since $f^{-1}(X_{j(M+1)})$ has at least $M+1$ components, each mapped *onto* $X_{j(M+1)}$ by f, the degree of the restriction of f to $f^{-1}(K_i - f(Z))$ is at least $M+1$, contradicting the definition of M.

Now, suppose that P satisfies C_2. If a simply connected region U_n meets a component K_i ($n = 1,2,\cdots$; $i = 1,2,\cdots,I$), then (cf. Condition C_2(a) and (2.2)) $K_i \subset U_n$. Since $\limsup(U_n) \subset P$, there exists M such that, if $n > M$, then

$$U_n \cap \bigcup_{i=1}^{I} K_i = \emptyset.$$

Since $P \cap R_m \neq \emptyset$ ($m = 1,2,\cdots$), there exists (by Condition C_2(a)), a region U_r such that $\mathrm{bdy}(U_r)$ meets R_m. Because

$$\mathrm{diam}(R_m) < \mathrm{diam}\, P,$$

it follows from Condition C_2(a) that there is a region U_s whose boundary meets $S - R_m$. By C_2(b) there are regions $U_{s(t)}$ ($t = 1,2,\cdots,T$) such that $s(1) = r$, $s(T) = s$, $U_{s(t)}$ meets $U_{s(t+1)}$ and $\mathrm{bdy}(U_{s(t)})$ meets $\mathrm{bdy}(U_{s(t+1)})$ ($t = 1,2,\cdots,T-1$). For some t, $U_{s(t)}$ meets $\mathrm{bdy}(R_m)$, and $\mathrm{bdy}(U_{s(t)})$ meets $\mathrm{bdy}(R_m)$. Let $s(t)$ be denoted by $n(m)$. There exists an arc γ_m on $\mathrm{bdy}(R_m)$ (a topological circle by 2.5) such that γ_m ends in a point q_m on $\mathrm{bdy}(U_{n(m)})$ and $\gamma_m - \{q_m\} \subset U_{n(m)}$ ($m = 1,2,\cdots$).

Since $R_1 \subset U$ and $\mathrm{bdy}(R_1) \cap \mathrm{bdy}\, U$ contains no arc, there exist open sets X_1 and Y_1, each meeting γ_1, such that

$$\bar{Y}_1 \subset X_1 \subset U \cap U_{n(1)}.$$

If $q_2 \in \bar{Y}_1$, choose open sets X_2 and Y_2, each meeting γ_2, such that

$$\bar{Y}_2 \subset X_2 \subset X_1 \cap U_{n(2)}.$$

(Thus $\bar{Y}_2 \subset U$.) In this case, since

$$q_2 \in U_{n(1)} \cap \mathrm{bdy}(U_{n(2)}),$$

$U_{n(1)} \neq U_{n(2)}$. If $q_2 \notin \bar{Y}_2$, choose open sets X_2 and Y_2, each meeting γ_2, such that

$$\bar{Y}_2 \subset X_2 \subset (U \cap U_{n(2)}) - \bar{Y}_1.$$

Using finite induction, suppose that open sets X_j and Y_j have been defined, each meeting γ_j, such that:

(1_m) $\bar{Y}_j \subset X_j \subset U$; and

(2_m) for each $i < j$, either $X_j \subset X_i \cap U_{n(j)}$ (in which case $U_{n(i)} \neq U_{n(j)}$), or $X_j \subset U_{n(j)} - Y_i$ ($j = 1,2,\cdots,m$).

Let V be an open neighborhood of q_{m+1} such that: if $q_{m+1} \notin \bar{Y}_j$, then $V \cap \bar{Y}_j = \emptyset$, and if $q_{m+1} \in \bar{Y}_j$, then $V \subset X_j$ ($j = 1,2,\cdots,m$). Now $V \cap U_{n(m+1)}$ meets γ_{m+1}, $\gamma_{m+1} \subset \bar{U}$, and $\gamma_{m+1} \cap \mathrm{bdy}\, U$ has no arc, so that $V \cap U_{n(m+1)} \cap U$ also meets γ_{m+1}. Let X_{m+1} be this open set, and let Y_{m+1} be an open set also meeting γ_{m+1}, such that $\bar{Y}_{m+1} \subset X_{m+1}$. The sets X_j and Y_j ($j = 1,2,\cdots,m+1$) satisfy (1_{m+1}) and (2_{m+1}).

Thus, there exist nonempty open sets X_m and Y_m satisfying conclusions (a), (b), and (c). For (d), suppose that, on the contrary, there exist increasing natural numbers $j(k)$ $(k = 1,2,\cdots, M+1)$ such that $X_{j(k+1)} \subset X_{j(k)}$; thus, by the second inductive condition, the regions $U_{n(j(k))}$ are distinct. Since $X_{j(M+1)}$ meets $\mathrm{bdy}(R_{j(M+1)})$ and $X_{j(M+1)} \subset U$, $X_{j(M+1)}$ meets $U \cap \mathrm{bdy}(R_{j(M+1)})$; by the definition of ε-region (2.2), $X_{j(M+1)}$ meets some component K_i $(i = 1,2,\cdots,I)$. As a result, each $X_{j(k)}$ meets K_i. Since $X_{j(k)} \subset U_{n(j(k))}$ (by the second inductive condition), the distinct sets $U_{n(j(k))}$ each meet, and thus contain K_i $(k = 1,2,\cdots,M+1)$, contradicting the choice of M.

3.2. Lemma. *Let P, U, ε, and R_m $(m = 1,2,\cdots)$ satisfy the hypotheses of* 3.1. *Then there exist simply connected regions L_i^h $(i = 1, 2, \cdots, I$, the integer I of* 2.2; *$h = 1,2,\cdots,M$, the integer M of* 3.1) *in U such that*:

(a) *each* $\mathrm{bdy}(L_i^h)$ *is a Peano continuum*;

(b) *there exists a function Ω mapping the indices of the ε-regions R_m into pairs of natural numbers (i,h) such that R_m meets L_i^h properly in a point q_m*;

(c) *the prime ends of L_i^h associated with these points by* 2.7, *are distinct, and no one of the prime ends is a limit of others.*

The limit statement of conclusion (c) means that: if $\bar{q}_m$ are the points on the circle $\mathrm{bdy}\, D$ given by the Carathéodory Theorem [2, p. 350] as corresponding to the prime ends, then no point $\bar{q}_m$ is a limit point of others.

Proof. Let X_m and Y_m $(m = 1,2,\cdots)$ be the open sets given by 3.1. Consider a fixed set R_m; by 2.5, $\bar{R}_m$ is a closed topological disk. By 3.1(b), there exists an arc α of the circle $\mathrm{bdy}(R_m)$ such that $\alpha \subset Y_m$ and (since $\mathrm{bdy}(R_m) \cap \mathrm{bdy}\, U$ contains no arc) $\alpha \cap \mathrm{bdy}\, U = \emptyset$. Since (by 2.2)

$$\mathrm{bdy}(R_m) \subset \bigcup_{i=1}^{I} \mathrm{bdy}(K_i),$$

the union of the closed sets $\alpha \cap \mathrm{bdy}(K_i)$ $(i = 1,2,\cdots,I)$ is α. Thus for some i (call it $i(m)$)

$$\dim(\alpha \cap \mathrm{bdy}(K_{i(m)})) = 1$$

[6, p. 30], so that $\alpha \cap \mathrm{bdy}(K_{i(m)})$ contains an arc β_m [6, p. 44]; let p_m be an interior point of β_m.

Choose η, $0 < \eta < 1/2m$ such that

$$S(p_m, \eta) \cap \mathrm{bdy}(R_m) \subset \beta_m$$

and $S(p_m,\eta) \subset Y_m$. Since $\mathrm{bdy}(K_{i(m)})$ is locally connected [25, p. 106, (2.2)], each point of β_m is regularly accessible from $K_{i(m)}$ [25, p. 111 and p. 112, (4.2)]. Choose the regular accessibility δ for η and p_m, $0 < \delta < \eta$. Let q be a point of the component of $\beta_m \cap S(p_m, \delta)$ containing p_m, $q \neq p_m$, and similarly choose ζ for $\eta - \delta$ and q, $\zeta + d(p_m, q) < \delta$; choose r in $S(q, \zeta) \cup K_{i(m)}$. There exist arcs ρ joining r to p_m in

$$(K_{i(m)} \cap S(p_m, \eta)) \cup \{p_m\},$$

and σ joining r to q in

$$(K_{i(m)} \cap S(p_m, \eta)) \cup \{q\}.$$

Since $\rho \cup \sigma$ has Property S, it is locally connected, and thus contains an arc τ joining p and q,

$$\tau - (\{p\} \cup \{q\}) \subset K_{i(m)} \cap S(p_m, \eta).$$

Since $\tau \subset S(p_m, \eta) \subset Y_m$, $\beta_m \subset \alpha \subset Y_m$, and $Y_m \subset U$, (3.1(a)), $\beta_m \cup \tau \subset U$.

Let C_m be the component of $U - (\beta_m \cup \tau)$ disjoint from R_m. Since $\bar{C}_m$ and $\bar{R}_m$ are closed topological disks (the latter by 2.5) whose intersection is an arc, their union is also a closed topological disk. Let E_m be the interior of $\bar{C}_m \cup \bar{R}_m$. Since $\bar{C}_m \subset S(p_m, \eta)$, $\bar{C}_m \subset Y_m$ and $\operatorname{diam}(\bar{C}_m) < 1/m$. Let q_m be a point on $\operatorname{bdy}(R_m) - \beta_m$; thus

$$q_m \in \operatorname{bdy}(E_m) \cup \operatorname{bdy}(K_{i(m)} \cup E_m).$$

Let γ_m be an arc ending at q_m, $\gamma_m - \{q_m\}$ in the open topological disk (by 2.5) R_m.

Define a function ω mapping the indices m of the sets R_m into the natural numbers by: $\omega(m) = h$ if and only if h is the maximal positive integer such that there exists a function $j(k)$ $(k = 1, 2, \cdots, h)$ with $j(k) < j(k+1)$, $j(h) = m$, and $X_{j(k+1)} \subset X_{j(k)}$ (h may be 1). Thus $h \leqq M$, the natural number given by 3.1 (d). Define $\Omega(m)$ as the ordered pair $(i(m), \omega(m))$ and let L_i^h be the union of K_i with those open topological disks E_m for which $\Omega(m) = (i, h)$ $(i = 1, 2, \cdots, I; h = 1, 2, \cdots, M)$.

Let E_m and E_n be two open topological disks added to some region K_i in forming L_i^h, i.e., $\Omega(m) = \Omega(n) = (i, h)$. Using the facts that $E_m \subset \bar{C}_m \cup R_m$ and $E_n \subset \bar{C}_n \cup R_n$ (R_m and R_n are open topological disks by 2.5), it will now be shown that $E_m \cap E_n = \emptyset$. Suppose that $Y_m \cap Y_n \neq \emptyset$; by conclusions (a) and (c) of 3.1, if $m < n$, then $X_n \subset X_m$, so that $\omega(m) < \omega(n)$. Since $\omega(m) = \omega(n) = h$, $Y_m \cap Y_n = \emptyset$. Thus, since $\bar{C}_m \subset Y_m$ and $\bar{C}_n \subset Y_n$, $\bar{C}_m \cap \bar{C}_n = \emptyset$. Moreover, the ε-regions R_m and R_n are disjoint, by Definition 2.2. Suppose $R_n \cap \bar{C}_m \neq \emptyset$. If $R_n \subset \bar{C}_m$, then, by conclusion (b) of (3.1), $Y_n \cap \bar{C}_m \neq \emptyset$, contradicting the fact that $Y_n \cap Y_m = \emptyset$. Thus, if $R_n \cap \bar{C}_m \neq \emptyset$, then R_n meets $\operatorname{bdy}(C_m)$. But

$$\operatorname{bdy}(C_m) \subset K_{i(m)} \cup \operatorname{bdy}(R_m),$$

and R_n does not meet either, by the definition of ε-region (2.2); thus, $R_n \cap \bar{C}_m = \emptyset$. Similarly $R_m \cap \bar{C}_n = \emptyset$, and, hence, E_m and E_n are disjoint.

Each set L_i^h is a region. Let Γ be any closed path in L_i^h, i.e., Γ is a map of the unit circle into L_i^h. The topological disks E_m for which $\Omega(m) = (i, h)$ are mutually disjoint, and each $\operatorname{bdy}(E_m)$ meets K_i in an open arc. There is a function $\Delta(t, x)$, where $t \in [0, 1]$ and x is on the unit circle, defined as follows: for $x \notin \Gamma^{-1}(E_m)$, $\Delta(t, z) = \Gamma(x)$; for $x \in \Gamma^{-1}(E_m)$, $\Delta(0, x) = \Gamma(x)$,

$$\Delta(t, x) \subset E_m \cup (K_i \cap \operatorname{bdy}(E_m)),$$

$$\Delta(1, x) \subset K_i \cap \operatorname{bdy}(E_m),$$

and the restriction map $\Delta \mid \Gamma^{-1}(\bar{E}_m)$ is continuous. We wish to prove that Δ is continuous, i.e., if $t_k \to t$ and $x_k \to x$, then $\Delta(t_k, x_k) \to \Delta(t, x)$. If x is in some set $\Gamma^{-1}(E_m)$, then all but a finite number of the x_k are in $\Gamma^{-1}(E_m)$, and the conclusion follows from the definition of Δ on $\Gamma^{-1}(E_m)$. Thus we may suppose that $x \notin \bigcup_m \Gamma^{-1}(E_m)$, so that $\Delta(t, x) = \Gamma(x)$, and that either (1) all $x_k \notin \bigcup \Gamma^{-1}(E_m)$ or (2) each x_k is in some set $\Gamma^{-1}(E_m)$, call it $\Gamma^{-1}(E_{m(k)})$. Since $\Delta \mid \Gamma^{-1}(\bar{E}_m)$ is continuous ($m = 1, 2, \cdots$), we may suppose in case (2) that the numbers $m(k)$ are distinct ($k = 1, 2, \cdots$). In case (1) $\Delta(t_k, x_k) = \Gamma(x_k)$, so that $\Delta(t_k, x_k) \to \Gamma(x_k)$. In case (2), $\Delta(t_k, x_k) \in \bar{E}_{m(k)}$. Since $\operatorname{diam}(C_m) \to 0$, $\operatorname{diam}(R_m) \to 0$ (by 2.4), and $\bar{E}_m = \bar{C}_m \cup \bar{R}_m$, $\operatorname{diam}(E_m) \to 0$. Since $\Gamma(x_k) \in \bar{E}_{m(k)}$, $\bar{E}_{m(k)} \to \Gamma(x)$, so that $\Delta(t_k, x_k) \to \Gamma(x)$. As a result Δ is continuous, so that Γ is homotopic in L_i^h to a closed path in

$$L_i^h - \bigcup \{E_m : \Omega(m) = (i, h)\},$$

a subset of K_i. Since K_i is simply connected (it is a component of $S - P$), this path is homotopic in L_i^h to a point. Thus, L_i^h is also a simply connected region, so that $\operatorname{bdy}(L_i^h)$ is a continuum.

Let $\xi > 0$ be given. Since K_i has Property S [25, p. 106, (2.2) and p. 112, (4.2)], K_i is the union of a finite number of connected subsets, each of diameter less than $\xi/3$. Since $\operatorname{diam}(E_m) \to 0$, only a finite number of the topological disks E_m have $\Omega(m) = (i, h)$ and $\operatorname{diam}(E_m) \geqq \xi/3$, and each of these sets is also the finite union of connected subsets of diameter less than $\xi/3$. The sets E_m such that $\Omega(m) = (i, h)$ and $\operatorname{diam}(R_m) < \xi/3$ each meet one of the connected subsets of K_i; thus, L_i^h is the union of a finite number of connected subsets, each of diameter less than ξ. Hence, L_i^h has Property S, and by [25, p. 112, (4.2)], $\operatorname{bdy}(L_i^h)$ is locally connected. Thus, conclusion (a) is satisfied.

We have seen that $E_m \cap E_n = \emptyset$ for $\Omega(m) = \Omega(n) = (i, h)$, and that

$$q_m \in \operatorname{bdy}(E_m) \cup \operatorname{bdy}(K_i \cup E_m);$$

thus $q_m \in \operatorname{bdy}(L_i^h)$, so that conclusion (b) is satisfied (using the arc γ_m). Also, because of the disjointness of the sets E_n, the open arc $\operatorname{bdy}(R_m) - \beta_m$ of $\operatorname{bdy}(K_i \cup E_m)$ is contained in $\operatorname{bdy}(L_i^h)$. Moreover, a neighborhood of one side of it is contained in $R_m \cap L_i^h$; thus, corresponding to each point of the open arc $\operatorname{bdy}(R_m) - \beta_m$ is a unique prime end "from R_m." Now q_m is on this open arc, and $\gamma_m - \{q_m\}$ is contained in the open topological disk R_m; because of the disjointness of the sets E_n, conclusion (c) follows.

3.3. Although the chains of prime end theory were used in the proof of 2.1, it is convenient to give a new concept the same name.

DEFINITION. A *chain* (for P) is a finite or countably infinite family of simply connected regions U_n in S ($n = 1, 2, \cdots$) such that:

(a) The space $P = \mathrm{Cl}\,[\bigcup_n \mathrm{bdy}(U_n)]$.

(b) Each $\mathrm{bdy}(U_n)$ is a nondegenerate Peano continuum.

(c) There is a function ψ sending the natural numbers greater than one into the natural numbers such that $\psi(n) < n$, and U_n meets $U_{\psi(n)}$ properly (2.7) in a point p_n. The accessibility arc defines a natural prime end P_n of $U_{\psi(n)}$ (see 2.7) whose impression is the point p_n.

(d) For each natural number k and the set of all natural numbers $n(m)$ ($m = 1, 2, \cdots$) such that $\psi_{(n(m))} = k$, the prime ends $P_{n(m)}$ (of U_k) are distinct and distinct from the point p_k, and no prime end $P_{n(m)}$ is a limit of others (see 3.2).

(e) Let

$$\Psi_n = \bigcup \{U_k : \psi^h(k) = n,\ h = 0, 1, 2, \cdots\},$$

where ψ^h is the hth iteration of ψ, and $\psi^0(k) = k$. If the family $\{U_n\}$ is infinite, then $\mathrm{diam}(\Psi_n) \to 0$. In particular, $\mathrm{diam}(U_n) \to 0$.

The *range* of the chain $\{U_n\}$ is $\bigcup_n U_n$. If P has a chain, then P satisfies condition C_2.

3.4. DEFINITIONS. Let f be a function of A into B, where A and B are topological spaces. If, whenever U is open in A, $f(U)$ is open in B, then f is *interior* (*open*). If, for each p in $f(A)$, the components of $f^{-1}(p)$ are single points, then f is *light*. It is well known that any nonconstant meromorphic function is light interior.

Let $f: D \to S$ be light interior with the Peano continuum P its global cluster set $C(f)$. Let $f = gh$ be the factorization given by [19, p. 121], where h is a homeomorphism and g is meromorphic. We may suppose that $h(D)$ is either D or $S - \{\infty\}$. Since P is neither S nor a single point, $h(D) = D$; hence P is the global cluster set $C(g)$, so that P satisfies Condition C_3. Thus Condition C_3 is a topological condition in the sense that P may be replaced by the set $h(P)$, where h is any homeomorphism of S onto itself.

As a result, the properties of the hypothesis and conclusion of the following lemma are preserved under a homeomorphism of S onto itself. In particular, D may be replaced by the interior E of any topological closed disk.

3.5. LEMMA. *Suppose that P satisfies either Condition C_2 or C_3, that $\varepsilon > 0$, that U is S or D, and, if U is D, then $u \in \mathrm{bdy}\,D$ and $\mathrm{bdy}\,D \subset P$. Then there exists a family $\{V_n\}$ such that*:

(1) *$\{V_n\}$ is a chain, except that, instead of Condition* (a), *each* $\mathrm{bdy}(V_n) \subset P$.

(2) *If $U = D$, then $V_1 = U$; if $\psi(n) = 1$, then $p_n \neq u$ ($n = 1, 2, \cdots$).*

(3) *Each $V_n \subset U$.*

(4) *There exists a natural number N such that, for all $n > N$:*

(a) *$\bar{V}_n$ is a closed topological disk*; *and*

(b) $\operatorname{diam}(V_n) < \varepsilon$.

(5) *For all* n, $\psi(n) \leqq N$.

(6) *For each point* p *in* $P \cap \bar{U}$, *either* $p \in \operatorname{bdy}(V_n)$ $(n = 1, 2, \cdots, N)$, *or* $p \in \bar{V}_n$ $(n > N)$.

If P *satisfies either Condition* C_2' *or* C_3', $\infty \notin P$, *and* $U = S$, *then* $\{V_n\}$ *may be chosen so that* $\infty \notin V_n$ $(n = 1, 2, \cdots)$.

Proof. If $U \subset P$, then, since $P \neq S$, $U = D$. Let h be a homeomorphism of the unit square I^2 onto $\bar{D}$, let δ be the uniform continuity δ for h and ε, and let m be a natural number such that $2^{-m} < \delta$. Let S_n $(n = 1, 2, \cdots)$ be the squares of the families $\mathfrak{D}_j$ (defined in 1.3; $j = 1, 2, \cdots$) taken in order, and let N be the index of the last square of side at most 2^{-m-1}. The sets V_n are $h(S_n)$.

Thus, it may be assumed that $U - P \neq \emptyset$, and that ε is less than diam P, $\frac{1}{2}$diam S and diam K, where K is some component of $U - P$. Let K_i $(i = 1, 2, \cdots, I)$ be the components of $U - P$ in 2.2, and let E_j $(j = 1, 2, \cdots, J)$ be the interiors of the closed topological disks given by 2.5 and 2.6 for the ε-regions (for P and U) of diameter at least $\varepsilon/3$.

Let $\mathfrak{P}$ be the family of those ε-regions R (for P and U) such that diam $R < \varepsilon/3$ and $R \cap P \neq \emptyset$. Each region R in $\mathfrak{P}$ is the interior of a closed topological disk (by 2.5). If $U = D$, let $\mathfrak{Q}$ be the family of those ε-regions R in $\mathfrak{P}$ such that bdy $R \cap$ bdy D contains an open arc, call it $\mu(R)$, and let $q(R) \in \mu(R)$. Then the open arcs $\mu(R)$ for R in $\mathfrak{Q}$ are disjoint, and, if L is the closure of $\bigcup\{q(R) : R \in \mathfrak{Q}\}$, then L contains no arc. If $U = D$, let $\mathfrak{R} = \mathfrak{P} - \mathfrak{Q}$. If $U = S$, let $\mathfrak{R} = \mathfrak{P}$. If $\mathfrak{R} \neq \emptyset$, let R_m be an enumeration of the family $\mathfrak{R}$, and let the regions L_i^h $(i = 1, 2, \cdots, I;\ h = 1, 2, \cdots, M)$, the function Ω, and the points q_m $(m = 1, 2, \cdots)$ be as given in 3.2.

If $U = D$, let $V_1 = D$; if $U = S$, let $V_1 = K_1$ (since $U - P \neq \emptyset$, $I \neq 0$). Let H be the number of regions K_i (other than K_1, if $V_1 = K_1$), L_i^h (there are none if $\mathfrak{R} = \{R_m\} = \emptyset$), and E_j $(i = 1, 2, \cdots, I;\ h = 1, 2, \cdots, M;\ j = 1, 2, \cdots, J)$. Let V_n $(n = H + 2, H + 3, \cdots, 2H + 1)$ be any enumeration of these sets with the sets E_j at the end, and let N be the index of the last region V_n not a topological disk E_j $(N = H + 1$ if all the sets are topological disks $E_j)$. We will now define simply connected regions V_n $(n = 2, 3, \cdots, H + 1)$ such that V_n meets both V_1 and V_{n+H} properly, $V_n \subset U$, $\operatorname{bdy}(V_n) \subset P$, and the points p_n of accessibility (3.3(c)) are distinct, different from u and from q_m $(m = 1, 2, \cdots)$, and in case $U = D$, not in L.

If $U = D$, then $V_1 = D$ and $V_{n+H} \subset D$; let $\lambda_1 = \operatorname{bdy} D$. If $U = S$, let λ_1 be an arc on $\operatorname{bdy}(V_1)$. If $\operatorname{bdy}(V_{n+H}) \subset \lambda_1$, let $\lambda_2 = \lambda_3 = \emptyset$. Otherwise, there exists an arc λ_2 disjoint from λ_1 and contained in $\operatorname{bdy}(V_{n+H})$. There is an arc λ_3 in P joining a point of λ_2 to a point of λ_1, λ_3 meeting $\lambda_1 \cup \lambda_2$ only in its endpoints. Let $V_n = U - (\lambda_1 \cup \lambda_2 \cup \lambda_3)$; then V_n is a simply connected subregion of U whose boundary $\lambda_1 \cup \lambda_2 \cup \lambda_3$ is a nonempty and nondegenerate Peano continuum contained in P. Since $\lambda_1 \subset \operatorname{bdy}(V_1) \cap \operatorname{bdy}(V_n)$ and $\lambda_2 \subset \operatorname{bdy}(V_n) \cap \operatorname{bdy}(V_{n+H})$, and since each point of bdy(V_1) (resp., bdy(V_{n+H})) is regularly accessible from V_1 (resp., V_{n+H}) [25, p. 112, (4.2)], V_1 (resp., V_{n+H}) meets V_n properly (2.7) in an arc of points of accessibility. Define $\psi(n)$ as 1, $\psi(n+H)$ as n, and p_n and p_{n+H} $(n = 2, 3, \cdots, H+1)$ as accessibility points, chosen so that they are distinct, none is u or q_m $(m = 1, 2, \cdots)$, and, in case $U = D$, none is in the 0-dimensional set L.

The regions V_n $(n = 2H+2,\ 2H+3, \cdots)$ are defined to be the open topological disks of $\mathfrak{P}$, i.e., of $\mathfrak{Q}$ and $\mathfrak{R}$, added (say) in order of decreasing diameter. If $V_n = R$ in $\mathfrak{Q}$, then $U = D = V_1$; let $\psi(n) = 1$, and let p_n be $q(R)$. Then V_n meets $V_1 = D$ properly in p_n. Suppose that $V_n = R_m$ in $\mathfrak{R}$; if $\Omega(m) = (i, h)$ and $V_k = L_i^h$, then let $\psi(n) = k$ and $p_n = q_m$.

Conclusions (1) (a), (b), (c) have already been noted or are immediate. For (1) (d) and for (2), observe that the points p_n $(n = 2, 3, \cdots, 2H+1)$ and u were defined to be distinct. Thus, to prove that (1) (d) and (2) are satisfied for all natural numbers k, it is sufficient to consider only those k for which there exists $n > 2H+1$ with $\psi(n) = k$. For each $n > 2H+1$, $V_n \in \mathfrak{Q} \cup \mathfrak{R}$. If $V_n \in \mathfrak{Q}$, then $\psi(n) = 1$ and $U = D = V_1$; conversely, $\psi(n) = 1$ only for V_n in $\mathfrak{Q}$ and for $n = 2, 3, \cdots, H+1$. Since $p_n = q(R)$ for $V_n = R$ in $\mathfrak{Q}$, conclusions (1)(d) and (2) follow for $k = 1$ from the definition of p_n $(n = 2, 3, \cdots, 2H+1)$. The set V_n is in $\mathfrak{R}$ (i.e., V_n is an open topological disk R_m) if and only if $\psi(n) = k$, where $V_k = L_i^h$ and $\Omega(m) = (i, h)$ $(k = 2, 3, \cdots)$; the point p_n is the point q_m, so that (1) (d) is satisfied for these k by 3.2(c) and by the choice of p_k.

Conclusions (3) and (5) follow immediately from the construction of $\{V_n\}$, and conclusion (4) from the fact that, for $n > N$, $\bar{V}_n$ is a closed topological disk of $\{E_j\}$ $(j = 1, 2, \cdots, J)$ or of $\mathfrak{P} = \mathfrak{Q} \cup \mathfrak{R}$. By conclusion (5) $\Psi_n = V_n$ (for $n > N$), and since all but a finite number of these sets are ε-regions, (1) (e) follows from 2.4. For (6), each point P in $P \cap \bar{U}$ is either in bdy$D$ (in case $U = D$, so that $V_1 = D$), in bdy(K_i) $(i = 1, 2, \cdots, I)$, or in some ε-region R (by 2.2). Thus it suffices to prove that for each ε-region R,

$$P \cap R \subset \bigcup_{n=N+1}^{\infty} V_n.$$

If diam$R \geqq \varepsilon/3$, then $R \subset \bigcup_{j=1}^{J} E_j$; if diam$R < \varepsilon/3$, and R meets P, then $R \in \mathfrak{P} = \mathfrak{R} \cup \mathfrak{Q}$. Conclusion (6) follows.

Lastly, suppose that P satisfies either Condition C_2' or C_3', $\infty \notin P$, and $U = S$. Let R be $\bigcup_n U_n$, or the range of f, as the case may be; then R is a region (in case C_2' use C_2'(b)) not containing ∞. Let K be the component of $S - P$ containing ∞; suppose that $K \cap R \neq \emptyset$. In case C_2' some region U_n $(n = 1, 2, \cdots)$ meets K. Since bdy$(U_n) \subset P$ and $\infty \notin U_n$, $\infty \notin \bar{U}_n$. Since $\infty \in K$, K meets bdy(U_n), and a contradiction results. In case C_3', while $f(D)$ meets K, $\infty \notin f(D)$ and $\infty \notin P$, which is the global cluster set $C(f)$. Thus $\infty \notin \mathrm{Cl}\,[f(D)]$, and there exists a point

$$q \in K \cap \mathrm{bdy}(\mathrm{Cl}\,[f(D)]).$$

Since $\mathrm{Cl}[f(D)] = f(D) \cup C(f)$, $C(f) = P$, and $P \cap K = \emptyset$, $q \in f(D)$. But $f(D)$ is open, contradicting the definition of q. Thus $K \cap R = \emptyset$ in both cases C_2' and C_3'.

Since bdy K is a Peano space [25, p. 106, (2.2)], and $P \subset \bar{R}$, bdy K is a simple closed curve C separating K from R [24, p. 114, (6.7)]. Let E be that open topological disk of $S - C$ containing R. Then $P \subset \bar{E}$ and $C \subset \mathrm{bdy}\ K \subset P$. Application of this lemma under the previous hypothesis (C_2 or C_3) to the case $U = E$ (see 3.4) yields a possibly finite sequence of regions $V_n \subset E$ by conclusion (3). Thus $\infty \notin V_n$ $(n = 1,2,\cdots)$.

A family $\{V_n\}$ satisfying conclusions (1) through (6) of the lemma is almost a chain, intuitively speaking, differing only by sets V_n of diameter less than ε. (Compare 3.3(a) with (1) and (6).) Each of these small sets V_n is a topological disk, and there exists a "finer almost-chain" for P and it. If one continues in this way, a chain for P is constructed; this is the idea of the next lemma.

3.6. LEMMA. *If P satisfies either Condition C_2 or C_3, then it has a chain $\{U_n\}$. If it satisfies either C_2' or C_3', then $\{U_n\}$ may be chosen so that its range does not contain ∞.*

Proof. Repeated applications of 3.5 will be used to define a chain $\{U_n\}$, where the function ψ^* and the accessibility points q_n are induced by that lemma. Define U_n as V_n for $U = S$ and $\varepsilon = 1$ $(n = 1,2,\cdots,N_1)$, where N_1 is the integer N of conclusion (4) of 3.5; define ψ^* (on $n = 2,3,\cdots,N_1$) as the function ψ and q_n as the point p_n of 3.5 (1) (c). Let $\mathfrak{E}_1$ be the family of those regions (interiors of closed topological disks, by 3.5(4)(a)) V_n such that $n > N_1$; for each region E in $\mathfrak{E}_1$, diam $E < 1$ (by (4)(b)). Let ω_1 be the function of $\mathfrak{E}_1$ into the natural numbers $1,2,\cdots,N_1$ such that: if $E = V_n$, then $\omega_1(E) = \psi(n)$ (see 3.5(1)(c) and (5)(a)); let q_E be p_n. In the final chain $\{U_n\}$ any region U_n such that $\psi^*(n) = k$ $(k = 1,2,\cdots,N_1)$ will be a topological disk E in $\mathfrak{E}_1$ with $\omega_1(E) = k$; and the points q_n will be the points q_E. As a result, for these numbers k, condition 3.3(d) will result from the corresponding condition, (1)(d), of 3.5. Eventually, each topological disk of $\mathfrak{E}_1$ will be included in $\{U_n\}$. (If $\mathfrak{E}_1 = \emptyset$, then $\{U_n\}$ $(n = 1,2,\cdots,N_1)$ is the entire chain.)

Enlarge the family U_n $(n = 1,2,\cdots,N_1)$ with those topological disks E of $\mathfrak{E}_1$ such that diam $E \geqq \frac{1}{2}$; by 3.5(1)(e) there are only a finite number (and there may be none). Let $I(1)$ be the subscript of the last region U_n, and let ψ^* and q_n be defined for these additional n (from ω_1 and the points q_E) as above.

There is no n $(n = 1,2,\cdots,I(1))$ such that $\psi^*(n) = k$ $(k = N_1 + 1, N_1 + 2,\cdots,I(1))$; indeed, there is no topological disk E in $\mathfrak{E}_1$ such that $\omega_1(E) = k$, for such a k (from 3.5(5)). Each region U_k is the interior of a closed topological disk (3.5(4)(a)), and $\mathrm{bdy}(U_k) \subset P$ (3.5(1)). For each such k, form the family $\{V_n\}$, call it $\{V_{n,k}\}$, given by 3.5 for $U = U_k$, $\varepsilon = 1/2$, and $u = p_k$. Add each region

$V_{n,k}$ $(n=2,3,\cdots,N(k)$, the N of 3.5(4)) as a set U_j, where $\psi^*(j)$ and q_j are induced by 3.5 in the natural way, i.e., if $U_j = V_{n,k}, \psi(n)$ (for $U = U_k$) is m, and $V_{m,k} = U_s$ $(V_{1,k} = U_k)$, then $\psi^*(j) = s$. Let N_2 be the index of the last region U_n thus far defined.

Let $\mathfrak{E}_2$ consist of the regions $V_{n,k}$ for $n > N(k)$ (3.5(4)), together with the regions E of $\mathfrak{E}_1$ other than U_n $(n = 1,2,\cdots,I_1)$. Let ω_2 be the function of $\mathfrak{E}_2$ into the natural numbers such that: on $\mathfrak{E}_1 \cap \mathfrak{E}_2$, ω_2 agrees with ω_1, and on $\mathfrak{E}_2 - \mathfrak{E}_1, \omega_2$ (with the points q_E) is induced by the function ψ of 3.5(1)(c), as was ω_1. In the final chain $\{U_n\}$ any region U_n such that $\psi^*(n) = k$ $(n > N_2; k = I(1)+1, I(1)+2, \cdots, N_2)$ will be a topological disk E in $\mathfrak{E}_2 - \mathfrak{E}_1$ with $\omega_2(E) = k$; for each such k, all such sets U_n result from a single application of 3.5 for U the interior U_m of a closed topological disk $(m = N_1 + 1,\ N_1 + 2, \cdots, I(1))$, $U_n \cup U_k \subset U_m$, and q_n will be q_E. Thus, condition 3.3(d) for such k results from the corresponding condition, (1)(d), of 3.5. Add to $\{U_n\}$ each disk E of $\mathfrak{E}_2$ with $\operatorname{diam} E \geqq 1/3$. Let $I(2)$ be the subscript of the last region U_n thus far defined, and, if $E = U_n$, let $\psi^*(n) = \omega_2(E)$ and $q_n = q_E$.

The mth stage in the construction is similar to the second, where $\mathfrak{E}_1$ and $\mathfrak{E}_2$ are replaced by $\mathfrak{E}_{m-1}$ and $\mathfrak{E}_m$, respectively, $\varepsilon = \frac{1}{2}$, by $\varepsilon = 1/m$; $I(1)$ by $I(m-1)$; and ω_1 and ω_2, by ω_{m-1} and ω_m, respectively.

A (possibly finite) sequence $\{U_n\}$ is thus defined; it will now be proved that $\{U_n\}$ is a chain. (A formal proof by induction is not given because it is so involved.) Since each region U_n $(n = 1,2,\cdots)$ arose from the use of 3.5, $\operatorname{bdy}(U_n)$ is a non-degenerate Peano continuum contained in P (3.5(1)); thus, 3.3(b) is satisfied. For each region E in $\mathfrak{E}_m$, $\operatorname{diam} E < 1/m$, and (from the use of 3.5(6))

$$P \subset \left[\bigcup_{n=1}^{I(m)} \operatorname{bdy}(U_n)\right] \cup \bigcup \{\bar{E} : E \in \mathfrak{E}_m\}$$

$(m = 1,2,\cdots)$. Since each region E in $\mathfrak{E}_m$ meets some region U_n properly $(n = 1,2,\cdots,I_m)$, $\bar{E}$ meets $\operatorname{bdy}(U_n)$; as a result $\bigcup_{n=1}^{I(m)} \operatorname{bdy}(U_n)$ is $(1/m)$-dense in P, so that

$$P \subset \operatorname{Cl}\left[\bigcup \operatorname{bdy}(U_n)\right],$$

and condition 3.3(a) is satisfied. Condition (c) of 3.3 follows from the construction (i.e., from the use of the corresponding condition, (1)(c), of 3.5).

Condition 3.3(d) has already been discussed (e.g., see the end of the first paragraph). Consider 3.3(e); let $(\psi^*)^h$ be the hth iteration of ψ^* $(h = 1,2,\cdots)$, and let $(\psi^*)^0$ be the identity function. For each $n > I(m)$, there exists a topological disk U_k in $\mathfrak{E}_m$, $k \leqq n$, such that for some h, $(\psi^*)^h(n) = k$ $(h = 0,1,2,\cdots)$. But $\Psi_k \subset U_k$, because these sets arose through successive applications of 3.5, first for U as U_k, then as subsets of U_k (from 3.5(3)). Since $U_k \in \mathfrak{E}_m$, $\operatorname{diam}(U_k) < 1/m$, $\operatorname{diam}(\Psi_k) < 1/m$ and 3.3(e) satisfied.

Suppose that P satisfies condition C_2' or C_3'. If $\infty \notin P$, then the $C_2' - C_3'$

version of 3.5 may be applied, so that none of the regions V_n defined in the first paragraph of this proof contains ∞; from the subsequent use of 3.5 and from 3.5(3), no U_n contains ∞.

Now suppose that $\infty \in P$. For each n with $\infty \notin U_n$, let $\lambda_n = \emptyset$. If $\infty \in U_1$, let λ_1 be any arc in P joining ∞ to a point r_1 of $\mathrm{bdy}(U_1)$, $\lambda_1 - \{r_1\} \subset U_1$. If $\infty \in U_n$ $(n = 2, 3, \cdots)$, and $\infty \notin U_{\psi^*(n)}$, then there is an arc α joining ∞ to a point x of $\mathrm{bdy}(U_n) \cup \mathrm{bdy}(U_{\psi(n)})$, $\alpha - \{x\} \subset U_n - \bar{U}_{\psi^*(n)}$. If $x \in \mathrm{bdy}(U_n)$, let $\beta = \emptyset$; otherwise, there is an arc $\beta \subset \mathrm{bdy}(U_{\psi^*(n)})$ joining x to a point y on $\mathrm{bdy}(U_n)$ (by 3.3(c)), $\beta - \{y\} \subset U_n$. Let $\lambda_n = \alpha \cup \beta$, and let r_n be its endpoint on $\mathrm{bdy}(U_n)$.

If $\infty \in U_n \cap U_{\psi^*(n)}$, we may suppose (by induction) that $\lambda_{\psi^*(n)}$ has been defined, $\lambda_{\psi^*(n)} - r_{\psi^*(n)} \subset U_{\psi^*(n)}$. If $\lambda_{\psi^*(n)}$ meets $\mathrm{bdy}(U_n)$, let λ_n be that subarc of it which contains ∞ and meets $\mathrm{bdy}(U_n)$ only in its other endpoint, call it r_n. Otherwise, there is an arc α in $\mathrm{bdy}(U_{\psi^*(n)})$ joining $r_{\psi^*(n)}$ to a point r_n on $\mathrm{bdy}\, U_n$ (by 3.3(c)), $\alpha - \{r_n\} \subset U_n$; let $\lambda_n = \alpha \cup \lambda_{\psi^*(n)}$. In any case, $U_{\psi^*(n)} \cap \lambda_n \subset \lambda_{\psi^*(n)}$.

Let $X_n = U_n - \lambda_n$, and let $W_{2n+1} = W_{2n} = X_n$; let $\psi(2n) = 2\psi^*(n) + 1$, and $\psi(2n+1) = 2n$ $(n = 1, 2, \cdots)$. That $\{W_m\}$ satisfies 3.3 except possibly for conditions (c) and (d), and that its range does not contain ∞ is immediate.

For each k and n $(k = 1, 2, \cdots; \psi^*(n) = k)$, X_n and X_k meet properly in q_n (in fact, in a prime end of X_k corresponding to Q_n in U_k) unless $\infty \in U_k$ (i.e., $\lambda_k \neq \emptyset$), $q_n = r_k$, $\lambda_k - \{r_k\} \not\subset U_n$ and $(\lambda_k - \{r_k\}) \cap U_n \neq \emptyset$. There is at most one such n, call it $M(k)$, and X_k meets $X_{M(k)}$ properly in a point $x_{M(k)}$ in $\lambda_k - \{r_k\}$. Let $p_{2n} = q_n$ for $n \neq M(k)$, and $p_{2M(k)} = x_{M(k)}$; let p_{2k+1} be any point on $\mathrm{bdy}(W_{2k+1})$ other than p_{2k} and the points p_m for $\psi^*(m) = 2k$. Since $W_{2k+1} = W_{2k}$, p_{2k+1} is a point of accessibility [25, p. 112, (4.2)]. Thus $\{W_m\}$ satisfies 3.3(c) and (d), and is a chain whose range does not contain ∞.

3.7. Corollary. *Condition* C_1 *implies* C_3, *and* C_3 *implies* C_2. *Also,* C_1' *implies* C_3' *and* C_3' *implies* C_2'.

Proof. The first and third implications are immediate from the definitions. The second and fourth follow from 3.6 and the definition of chain (3.3).

Thus, to complete the proof of 1.1, it suffices to prove that C_2 implies C_1, and C_2' implies C_1'.

4. Construction of the function.

4.1. Lemma. *Let* σ *be a closed topological disk, and let the map* $g: \sigma \to S$ *be an orientation-preserving homeomorphism on* $\mathrm{int}\,\sigma$. *Let* λ *be an arc on* $\mathrm{bdy}\,\sigma$, *and let* μ *be an arc in* $\mathrm{int}\,\sigma$, *except for its endpoints, one on* $(\mathrm{bdy}\,\sigma) - \lambda$, *the other,* O, *on* $\mathrm{int}\,\lambda$. *Lastly, let* U *be a simply connected region in* S *such that* $\mathrm{bdy}\,U$ *is a nonempty Peano continuum other than a single point, and* $g(\mu)$ *is an arc inside* U *except for an endpoint* $g(O)$ *on* $\mathrm{bdy}\,U$. *Then there exists a topological closed disk* τ *in* σ *and a map* h *of* σ *onto* $g(\sigma) \cup U$ *such that:*

(1) *$\tau \cap \operatorname{bdy} \sigma$ is a subarc v of* $\operatorname{int} \lambda$, *and* $O \in \operatorname{int} v$;

(2) *h agrees with g on a neighborhood* (*in* σ) *of* $\operatorname{bdy} \sigma - \operatorname{int} \lambda$;

(3) *h is* 2-*to*-1 *interior on* $\operatorname{int} \sigma$;

(4) *h on τ is topologically equivalent to a map on $\bar{D}$ which on D is a conformal homeomorphism onto $U - g(\omega)$, where ω is a subarc of μ*;

(5) $h(\mathrm{Cl}[\lambda - \tau]) = g(\lambda)$; *and*

(6) $h(\lambda \cap \operatorname{bdy} \tau) = \operatorname{bdy} U$.

Proof. We may as well assume that σ is the triangle with vertices in rectangular coordinates $(0,2)$, $(-3,0)$, and $(3,0)$, that λ is its (horizontal) base, and that μ is its altitude, where O is the origin. Let P be $(0,1)$, Q_1 be $(-2,0)$, Q_2 be $(2,0)$, R_1 be $(-1,0)$, and R_2 be $(1,0)$; let τ be the triangle whose vertices are P, R_1, and R_2. Outside of the open triangle whose vertices are P, Q_1, and Q_2, let h agree with g; thus (1) and (2) are satisfied.

Let ω be that subarc of μ having endpoints P and O. The set $U - g(\omega)$ is a simply connected region, and its boundary is a Peano continuum (since it is the union of two sets having Property S [25, p. 20]). There exists a conformal homeomorphism of D onto $U - g(\omega)$, and its extension j to $\bar{D}$ is continuous (by (2.1)). The circle $\operatorname{bdy} D$ is the union of three arcs a, b, and c which meet one another only in their endpoints, a and b each mapped homeomorphically onto $g(\omega)$, a on one side (in the sense of prime ends of $U - g(\omega)$), b on the other side. There exists an orientation-preserving homeomorphism of $\operatorname{bdy} \tau$ onto $\operatorname{bdy} D$, sending the side PR_1 onto one of these two arcs, and the side PR_2 onto the other. This map may be extended to a homeomorphism k of τ onto $\bar{D}$, by sending concentric triangles onto concentric circles in the corresponding way. On τ, h is defined as jk; thus (4) and (6) are satisfied. If we imagine that $g(\omega)$ is vertical, with $g(O)$ at the bottom, then h sends the right side of PR_1 onto the right side of $g(\omega)$, and the left side of PR_2 onto the left side of $g(\omega)$.

Let m be any orientation-preserving homeomorphism of the boundary of the triangle PQ_1R_1 onto the boundary of PQ_1O which is the identity map on PQ_1 and $g^{-1}h$ on PR_1. Let n be the homeomorphism of the triangles themselves, defined by sending concentric triangles onto concentric triangles in the corresponding way. On the triangle PQ_1R_1 the map h is defined as gn, and on PQ_2R_2 it is defined analogously. Conclusion (5) and the fact that h is continuous follow. Since g and n are orientation-preserving, h maps the left side of PR_1 onto the left side of $g(\omega)$, and the right side of PR_2 onto the right side of $g(\omega)$. Thus there is no folding along the straight line segments PQ_1, PQ_2, PR_1 and PR_2, and hence, h is a local homeomorphism except at P. As a result, h is interior [26, p. 82, (2.3)], yielding conclusion (3). (In fact, h locally at P is topologically equivalent to the analytic function $f(z) = z^2$.)

4.2. Lemma. *If P has a chain $\{U_n\}$, then there exists a map $f: \bar{D} \to S$*

such that f is light interior on D, $f(D)$ is contained in the range of $\{U_n\}$, and $f(\text{bdy}D) = P$.

Proof. There exists a conformal homeomorphism of D onto U_1, and its extension f_1 to $\bar{D}$ is continuous (by 2.1). If U_1 is the only region of the chain, let f be f_1.

Otherwise, let γ_n and p_n $(n = 2,3,\cdots)$ be the accessibility arc and point for U_n and $U_{\psi(n)}$ (given by 3.3 (c)). For each n such that $\psi(n)= 1$, $\mathrm{Cl}\,[f_1^{-1}(\gamma_n - \{p_n\})]$ is an arc η_n in $\bar{D}$ meeting bdy D in a single point O_n (by 2.7). The points O_n correspond to the prime ends P_n of 3.3 (c) under the Carathéodory correspondence, i.e., $f(O_n) = p_n$, so that no one of the points O_n is a limit point of others (3.3 (d)). Thus there are subarcs μ_n of η_n ending in O_n which are mutually disjoint (for n with $\psi(n) = 1$); we may also suppose that $\mathrm{diam}(\mu_h) < 1/n$ and that $\mathrm{diam}\,(f_1(\mu_n)) < 1/n$. Thus for each n and k such that $\psi(n) = \psi(k)$, either $p_n \neq p_k$ and $f(\mu_n) \cap f(\mu_k) = \emptyset$ or $p_n = p_k$ and $f(\mu_n) \cap f(\mu_k)$ is the point p_n.

For each n such that $\psi(n) = 1$, there exists a (closed) topological disk σ_h in $\bar{D}$ such that:

(1) each set $\sigma_n \subset D$, except for an arc λ_n on bdy$D$ containing O_n in its interior;
(2) each arc $\mu_n \subset \mathrm{int}(\sigma_n)$, except for its ($\mu_n$'s) endpoints, O_n and one in D;
(3) $\mathrm{diam}(\sigma_n) < 1/n$, and (by the continuity of f_1)

$$f_1(\sigma_n) \subset S(p_n, 1/n);$$

(4) the topological disks σ_n are mutually disjoint; and
(5) no one of them contains a limit point of others.

Let f_2 be the map given by: f_2 agrees with f_1 off $\mathrm{int}(\sigma_2)$, and on σ_2 is the map h given by 4.1, where g is the restriction of f_1 to σ_2, $\lambda = \lambda_2$, $\mu = \mu_2$, and $U = U_2$. Then (by 4.1 (2) and (3)) f_2 is finite-to-one interior on D, $f_2(D)$ is $U_1 \cup U_2$, and (by 4.1 (2), (5) and (6))

$$f_2(\text{bdy}\, D) = \text{bdy}\,(U_1) \cup \text{bdy}\,(U_2).$$

(This construction of f_2 is for clarity only, and it will not be used hereafter.)

Now, suppose that maps $f_i : \bar{D} \to S$ $(i = 1,2,\cdots,m)$ have been defined such that:

(1) For each n such that $\psi(n) = 1,2,\cdots,m$ there is a closed topological disk σ_n contained in D except for an arc λ_n on bdy D.

(2) (a) There is an arc μ_n such that the set $f_{\psi(n)}(\mu_n)$ is a subarc of γ_n.

(b) Moreover, $\mu_n \subset \mathrm{int}(\sigma_n)$, except for its two endpoints, one in D, the other, O_n, in $\mathrm{int}(\lambda_n)$; and $f_{\psi(n)}(O_n) = p_n$.

(3) Let $\mathfrak{S}_i$ $(i = 1,2,\cdots,m)$ be the set of topological disks σ_n, where $n > i$ and $\psi(n) \leqq i$.

(a) The topological disks of $\mathfrak{S}_i$ are disjoint, and no one of them has a limit point of others.

(b) If $j < i$ then for each topological disk σ_n of $\mathfrak{S}_i$, either $\sigma_n \in \mathfrak{S}_j$, or there exists a topological disk σ_k in $\mathfrak{S}_j$ such that

$$\sigma_n \subset \text{int}(\sigma_k) \cup \text{int}(\lambda_k).$$

(c) Given two distinct topological disks σ_n and σ_k in $\bigcup_{i=1}^m \mathfrak{S}_i$, then $\sigma_n \subset \sigma_k$ if and only if $\psi^h(n) = k$ $(h = 1,2,\cdots)$.

(d) Furthermore, f_i agrees with f_{i-1} on $\text{Cl}[D - \sigma_i]$ $(i = 2,3,\cdots,m)$.

(4) For each topological disk σ_n in $\mathfrak{S}_i$, f_i is a homeomorphism on $\text{int}(\sigma_n)$ $(i = 1,2,\cdots,m)$.

(5) For each topological disk σ_n in $\bigcup_{i=1}^m \mathfrak{S}_i$ and j $(j = 1,2,\cdots,m)$:

(a) $\text{diam}(\sigma_n) < 1/n$, and

(b) $f_j(\sigma_n) \subset S(p_n, 1/n) \cup \Psi_n$ (see 3.3(e)).

(6) (a) On D, f_i is finite-to-one interior,

(b) $f_i(D) \subset \bigcup_{j=1}^i U_j$, and

(c) $f_i(\text{bdy}\, D) = \bigcup_{j=1}^i \text{bdy}(U_j)$ $(i = 1,2,\cdots,m)$.

Call these six conditions Property $\mathfrak{Q}_m$ of the set of functions $f_1,f_2,\cdots,f_m$. Observe that f_1 satisfies $\mathfrak{Q}_1$ (we will not give a separate proof that the set f_1,f_2 satisfies $\mathfrak{Q}_2$). If the chain $\{U_n\}$ consists of more than m regions, we will now construct a function f_{m+1} such that the set $f_1,f_2,\cdots,f_{m+1}$ has Property $\mathfrak{Q}_{m+1}$.

Each topological disk $\sigma_k \in \mathfrak{S}_{k-1}$ $(k = \psi(m + 1) + 1,\ \psi(m + 1) + 2,\cdots,m)$; since $\sigma_{m+1} \in \mathfrak{S}_{k-1}$, $\sigma_{m+1} \cap \sigma_k = \emptyset$ (by $\mathfrak{Q}_m$, (3)(a)). Thus f_m agrees with $f_{\psi(m+1)}$ on σ_{m+1} (by $\mathfrak{Q}_m$, (3)(d)), so that $f_m(\mu_{m+1})$ is a subarc of γ_{m+1} and $f_m(O_{m+1}) = p_{m+1}$ (by $\mathfrak{Q}_m$, (2)).

The map f_{m+1} will agree with f_m off the closed topological disk σ_{m+1}. Let g be the restriction of f_m to this topological disk; f_{m+1} on σ_{m+1} is the map h given by 4.1 for $\lambda = \lambda_{m+1}$, $\mu = \mu_{m+1}$, and $U = U_{m+1}$; let τ be the topological closed disk also given by 4.1. It follows that f_{m+1} is continuous, and that conditions $\mathfrak{Q}_{m+1}$, (3)(d), (6)(a) and (6)(b) are satisfied (by inductive hypothesis, and 4.1(2) and (3)).

By inductive hypothesis, condition $\mathfrak{Q}_{m+1}$, (5)(b) is satisfied except for the cases $j = m + 1$ or $\psi(n) = m + 1$. Suppose that $j = m + 1$ and $\psi(n) = 1,2,\cdots,m$. If $\sigma_n \cap \sigma_{m+1} = \emptyset$, then, since f_m agrees with f_{m+1} off σ_{m+1}, $\mathfrak{Q}_{m+1}$, (5)(b) again follows from the inductive hypothesis. Thus we may suppose that $\sigma_n \cap \sigma_{m+1} \neq \emptyset$. Since $\sigma_n \in \mathfrak{S}_{\psi(n)}$, $\sigma_{m+1} \in \mathfrak{S}_{\psi(m+1)}$, $\psi(n) \leqq m$, and $\psi(m + 1) \leqq m$, it follows from conditions $\mathfrak{Q}_m$, (3)(a), (b), and (c) that $\psi^h(m + 1) = n$ $(h = 1,2,\cdots)$ and $\sigma_{m+1} \subset \sigma_n$. Now since $f_m = f_{m+1}$ off σ_{m+1},

$$f_{m+1}(\sigma_n) \subset f_m(\sigma_n) \cup f_{m+1}(\sigma_{m+1}).$$

By 4.1,

$$f_{m+1}(\sigma_{m+1}) \subset f_m(\sigma_{m+1}) \cup U_{m+1} \subset f_m(\sigma_m) \cup \overline{\Psi}_n.$$

As a result, by the inductive hypothesis,

$$f_{m+1}(\sigma_n) \subset S(p_n, 1/n) \cup \overline{\Psi}_n.$$

Thus, to complete the proof that condition $\mathfrak{Q}_{m+1}$, (5)(b) is satisfied, we need only define the sets σ_n such that $\psi(n) = m + 1$, and prove that the condition is satisfied for $\psi(n) = m + 1$.

By 4.1(4), f_{m+1} on τ is topologically equivalent to a map on $\bar{D}$ which on D is a conformal homeomorphism onto $U_{m+1} - \alpha$, where α is a subarc of γ_{m+1} containing p_{m+1}. For each n such that $\psi(n) = m + 1$, $p_n \neq p_{m+1}$ (3.3(d)), so that we may suppose that $\gamma_n \cap \alpha = \emptyset$. The set

$$\mathrm{Cl}[(\mathrm{int}\,\tau) \cap f_{m+1}^{-1}(\gamma_n)]$$

is an arc η_n which meets $\mathrm{bdy}\, D \cap \mathrm{bdy}\, \tau$ in a single point O_n (by 2.7), and $f_{m+1}(O_n) = p_n$. Since the points O_n with $\psi(n) = m + 1$ are disjoint and no one is a limit point of others (3.3(d)), there is a subarc μ_n of η_n containing O_n such that the arcs μ_n are mutually disjoint, $\mathrm{diam}(\mu_n) < 1/n$, and $\mathrm{diam}(f_j(\mu_n)) < 1/n$ $(j = 1,2,\cdots,m + 1)$. Thus $\mathfrak{Q}_{m+1}$, (2)(a) is satisfied.

For each n such that $\psi(n) = m + 1$, there exists a closed topological disk σ_n in τ such that:

(1) Each $\sigma_n \subset \mathrm{int}\,\tau$ except for an arc $\lambda_n \subset \mathrm{int}(\lambda_{m+1})$, and $O_n \in \mathrm{int}(\lambda_n)$. (Thus $\mathfrak{Q}_{m+1}$, (1) is satisfied.)

(2) The arc $\mu_n \subset \mathrm{int}(\sigma_n)$ except for its endpoints, one in $\mathrm{int}\,\tau$, the other O_n (thus $\mathfrak{Q}_{m+1}$, (3)(b) is satisfied).

(3) Each $\mathrm{diam}(\sigma_n) < 1/n$ (so that $\mathfrak{Q}_{m+1}$, (5)(a) is satisfied), and each $f_j(\sigma_n) \subset S(p_n, 1/n)$ $(j = 1,2,\cdots,m + 1;\ \psi(n) = m + 1;$ yielding the remainder of $\mathfrak{Q}_{m+1}$, (5)(b)).

(4) The disks σ_n are mutually disjoint.

(5) No one of them contains a limit point of others.

Let $\mathfrak{S}_{m+1}$ consist of the topological closed disks just defined, together with disks of $\mathfrak{S}_m - \{\sigma_{m+1}\}$. Condition $\mathfrak{Q}_{m+1}$, (3)(a) follows from (4) and (5) above (and, of course, the inductive hypothesis); and $\mathfrak{Q}_{m+1}$, (3)(b) from (1) since $\tau \subset \sigma_{m+1}$. Consider condition $\mathfrak{Q}_{m+1}$, (3)(c). For $\psi(n) \leqq m$ and $\psi(k) \leqq m$, it follows by inductive hypothesis; for $\psi(n) = m + 1 = \psi(k)$, it follows from (4) above. Suppose $\psi(n) = m + 1$ and $\psi(k) \leqq m$. Then $\sigma_n \subset \sigma_{m+1}$, and (by $\mathfrak{Q}_m$,(3)(a),(b), and (c)) either $\sigma_k \cap \sigma_{m+1} = \emptyset$, or $\sigma_{m+1} \subset \sigma_k$ and $\psi^h(m + 1) = k$ $(h = 1,2,\cdots)$. Thus condition $\mathfrak{Q}_{m+1}$, (3)(c) is satisfied.

Each topological disk σ_n of $\mathfrak{S}_{n+1}$ either has $\psi(n) \leqq m$ so that $\sigma_n \in \mathfrak{S}_m$ and $\sigma_n \cap \sigma_{m+1} = \emptyset$ (by $\mathfrak{Q}_m$, (3)(a)), or $\psi(n) = m + 1$ and $\mathrm{int}(\sigma_n) \subset \mathrm{int}\,\tau$. Since f_{m+1} agrees with f_m off σ_{m+1}, and since f_{m+1} on $\mathrm{int}\,\tau$ is a homeomorphism (by 4.1(4)), condition $\mathfrak{Q}_{m+1}$, (4) follows.

The map f_{m+1} agrees with f_m off σ_{m+1}, so that the restriction maps $f_{m+1} \mid \mathrm{bdy}\, D$ and $f_m \mid \mathrm{bdy}\, D$ agree off λ_{m+1} (see (1)). By the construction of f_{m+1} and by 4.1(5) and (6),

$$f_{m+1}(\mathrm{Cl}[\lambda_{m+1} - \tau]) = f_m(\lambda_m)$$

and

$$f_{m+1}(\lambda_m \cap \operatorname{bdy} \tau) = \operatorname{bdy}(U_{m+1}).$$

Condition $\mathfrak{Q}_{m+1}$, (6)(c) then follows from the induction hypothesis. Thus, the set of functions $f_1, f_2, \cdots, f_{m+1}$ satisfies condition $\mathfrak{Q}_{m+1}$.

If the chain $\{U_n\}$ is finite, consisting of m regions, let f be f_m; it has the desired properties by $\mathfrak{Q}_m$, (6). If the chain is infinite, then, by the induction, there exists a sequence of functions f_m; the function f will now be defined. Let p be any point of D, and let M be such that $1/M < d(p, \operatorname{bdy} D)$. Then for all $m \geqq M$, the maps f_m agree (by $\mathfrak{Q}_m$, (3)(d) and (5)(a)) and are finite-to-one interior (by $\mathfrak{Q}_m$, (6)(a)) on $\{z : |z| < 1 - 1/M\}$, which contains p. The map f is defined on this set to agree with these maps; thus the restriction map $f|D$ is defined and is light interior. It follows from $\mathfrak{Q}_m$, (6)(b) that $f(D)$ is contained in the range of U_n.

We will prove that the maps $f_m|D$ converge uniformly to $f|D$. Given $\varepsilon > 0$, choose M ($M = 1, 2, \cdots$) such that $1/M < \varepsilon/4$ and $\operatorname{diam}(\Psi_m) < \varepsilon/4$ (by 3.3(e); $m = M, M+1, \cdots$). Since $f|D$ agrees with $f_m|D$ except on $\bigcup_{n=m+1}^{\infty} \operatorname{int}(\sigma_n)$ (by $\mathfrak{Q}_m$, (3)(d)), it suffices to prove that, for each z in $\operatorname{int}(\sigma_n)$, $d(f_m(z), f(z)) < \varepsilon$ ($m = M, M+1, \cdots; n = M+1, M+2, \cdots$). Now

$$f_m(\sigma_n) \subset S(p_n, 1/n) \cup \overline{\Psi}_n$$

(by $\mathfrak{Q}_m$, (5)(b)), and $p_n \in \bar{U}_n \subset \overline{\Psi}_n$ (by 3.3(e)), so that $\operatorname{diam}(f_m(\sigma_n)) < \varepsilon/2$. Let $y \in D \cap \operatorname{bdy}(\sigma_n)$; then $f_m(y) = f(y)$ ($m = M, M+1, \cdots$). There exists $k \geqq M$ such that $f_k(z) = f(z)$. Since $d(f_k(z), f_k(y)) < \varepsilon/2$, $d(f_m(z), f_m(y)) < \varepsilon/2$, and $f_k(y) = f_m(y)$, it results that $d(f_m(z), f(z)) < \varepsilon$. Thus the maps $f_m|D$ converge uniformly to $f|D$.

Since $\bar{D}$ is compact, each map f_m is uniformly continuous; it follows that $f|D$ is uniformly continuous. Thus $f|D$ has a unique continuous extension to $\bar{D}$, call it f, and the maps f_m converge uniformly to f.

For each n and m ($n = 1, 2, \cdots; m = n, n+1, \cdots$) and point $w \in \operatorname{bdy}(U_n)$ there is a point $z_m \in \operatorname{bdy} D$ such that $f_m(z_m) = w$ (by $\mathfrak{Q}_m$, (6)(c)). The sequence $\{z_m\}$ has a subsequence $\{z_{m(k)}\}$ ($k = 1, 2, \cdots$) convergent to a point z on $\operatorname{bdy} D$. Since the functions $f_{m(k)}$ converge uniformly to f, and the points $f(z_{m(k)})$ converge to $f(z)$, it follows that the points $f_{m(k)}(z_{m(k)})$ converge to $f(z)$; thus $f(z) = w$. As a result, $\operatorname{bdy}(U_n) \subset f(\operatorname{bdy} D)$, so that

$$\operatorname{Cl}\left[\bigcup_{n=1}^{\infty} \operatorname{bdy}(U_n)\right] \subset f(\operatorname{bdy} D).$$

By $\mathfrak{Q}_m$, (3)(a) and (b), and (5)(a),

$$\operatorname{bdy} D - \bigcap_{m=1}^{\infty} \bigcup_{n=m}^{\infty} \sigma_n$$

is dense in $\operatorname{bdy} D$. Since, by $\mathfrak{Q}_m$, (3)(d) and (6)(c),

$$f\left(\operatorname{bdy} D - \bigcap_{m=1}^{\infty} \bigcup_{n=m}^{\infty} \sigma_n\right) \subset \bigcup_{n=1}^{\infty} \operatorname{bdy}(U_n),$$

it follows that

$$f(\operatorname{bdy} D) \subset \operatorname{Cl}\left[\bigcup_{n=1}^{\infty} \operatorname{bdy}(U_n)\right].$$

4.3. LEMMA. *Condition* C_2 *implies* C_1, *and* C_2' *implies* C_1'.

Proof. If P satisfies condition C_2 (respectively, C_2'), then, from 3.6 and 4.2, there exists a continuous function $f:\bar{D}\to S$, f light interior on D, such that $f(\operatorname{bdy} D) = P$ (respectively, and $\infty \notin f(D)$). Let gh be the factorization given by [3], where h is continuous on $\bar{D}$ and a homeomorphism on D, and g is continuous on $h(\bar{D})$ and meromorphic (respectively, analytic) on $h(D)$. The functions h and g are extensions of the functions H and G, respectively, defined in the first paragraph of that proof, and thus $h(D)$ may be supposed to be either D or the complement in S of a single point. In the latter case, since $h(\operatorname{bdy} D)$ and $g(\operatorname{bdy} D)$ would be single points, $f(\operatorname{bdy} D)$ would be also; but it was assumed in 1.3 that P, which is $f(\operatorname{bdy} D)$, is neither S nor a single point. Thus, $h(D) = D$, and g is continuous on $\bar{D}$ and meromorphic (respectively, analytic) on D; as a result, P satisfies condition C_1 (respectively, C_1').

Theorems 1.1 and 1.2 follow from 3.7 and 4.3.

5. **Concluding remarks.**

5.1. Condition C_2 is topological in the sense that: if P is a Peano continuum in S, and h is a homeomorphism of S onto itself, then P satisfies C_2 if and only if $h(P)$ does. It is not sufficient that h be a homeomorphism on P alone; i.e., the embedding is significant. Let P be the example of [4], modified by replacing each closed disk by its bounding circle and a radius segment; P does not satisfy condition C_1 (see the first remark of §2 of [4]). Let P' be P, except that each radius segment is turned outside the disk; then P' is the boundary of a simply connected region, but P and P' are homeomorphic.

One might ask: can D in 1.1 be replaced by an arbitrary simply connected region U whose boundary is a nonempty and nondegenerate Peano continuum? In this case C_1 implies C_2. For the proof, let f be the function meromorphic on U such that $f(\operatorname{bdy} U) = P$. There exists a continuous function $g:\bar{D}\to\bar{U}$, such that g is a meromorphic homeomorphism on D (by the Riemann Mapping Theorem and 2.1), and $P = (fg)(\operatorname{bdy} D)$. Thus, 1.1 can be applied. On the other hand, condition C_2' does not imply C_1' in this case. Let λ be a radius segment in D, let $U = D - \lambda$, and suppose that there exists a continuous function $f:\bar{U}\to\bar{D}$ with f analytic on U and $f(\operatorname{bdy} U) = \operatorname{bdy} D$. From the openness of f and the fact that $f(U) \subset D$ it follows that f on $\operatorname{bdy} U$ as a function of the prime ends (whose impressions are single points) is either sense-preserving or -reversing. As a result, f could not be continuous on bdy U; thus U does not satisfy condition C_1'.

Next, consider the following question: for what functions α: bdy $D \to S - \{\infty\}$ is there a continuous function $f:\bar{D} \to S$ with $f \mid D$ analytic and $f \mid \text{bdy}\, D = \alpha$? Morse [9] and Titus [21 and 22] considered this question for the special case of functions α with continuous and nonzero first derivative. Morse gave a sufficient condition for a function α to be a boundary function in this sense, and Titus considered a subclass called normal representations and gave a necessary and sufficient condition for such functions to be boundary functions. Related results are given in Titus [20] and Titus and Young [23].

For the general question (no assumption on the differentiability of α), a necessary condition in terms of the range of α is, of course, given by 1.1. Also, by Riesz-Nevanlinna Theorem [10, p. 19] either α is a constant function, or α is constant on no arc. More generally, let $f:\bar{D} \to S$ be continuous on $\bar{D}$, light open on D, with $f(\text{bdy}\, D)$ neither S nor a single point; let gh be the factorization of 4.3, and let α, β, and γ be the restrictions of f, h and g to bdy D, respectively. Then β is a sense-preserving (or -reversing) map of the circle onto itself, and thus differs from a homeomorphism only in that some point inverses may be arcs. Since no point inverse of α is an arc, β (and thus h) is a homeomorphism. To within topological equivalence, then, the question is reduced to a topological question (which still seems imposing), namely: What is the class of boundary functions α for light open maps, where α maps no arc onto a single point?

In [21] and [22] Titus used a special case of this approach. He proved that for each normal representation α in his class, there is a map $f:\bar{D} \to S - \{\infty\}$, f light open on D and a local homeomorphism on a neighborhood of bdy D. As he observed, in this case the above factorization follows directly from the theorems of Stoilow [19, p. 121] and Carathéodory [1, p. 86], where h is a homeomorphism of $\bar{D}$ onto $\bar{D}$.

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